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N Abarenkova†‡, J-Ch Anglès d'Auriac§, S Boukraa‖¶, S Hassani[+] and J-M Maillard‖

† Centre de Recherches sur les Très Basses Températures, B P 166, F-38042 Grenoble, France
‡ Theoretical Physics Department, St Petersburg State University, Ulyanovskaya 1, 198904 St Petersburg, Russia
§ Centre de Recherches sur les Très Basses Températures, B P 166, F-38042 Grenoble, France
‖ LPTHE, Tour 16, 1er étage, 4 Place Jussieu, 75252 Paris Cedex, France
¶ Institut d'Aéronautique, Université de Blida, BP 270, Blida, Algeria
[+] CDTN, Boulevard F Fanon, 16000 Alger, Algeria

E-mail: `dauriac@crtbt.polycnrs-gre.fr`, `hassani@ist.cerist.dz` and `maillard@lpthe.jussieu.fr`

Abstract. We consider a family of birational measure-preserving transformations of two variables, depending on one parameter, for which simple rational expressions with integer coefficients, for the exact expression of the dynamical zeta function, have been conjectured, together with an equality between the (multiplicative rate of growth of the) Arnold complexity and the (exponential of the) topological entropy. This identification takes place for the birational mapping seen as a mapping bearing on two complex variables (acting in a complex projective space). We revisit this identification between these two quite 'universal complexities' by considering now the mapping as a mapping bearing on two real variables. The definitions of the two previous 'topological' complexities (Arnold complexity and topological entropy) are modified according to this real-variables point of view. Most of the 'universality' is lost. However, the results presented here are, again, in agreement with an identification between the (multiplicative rate of growth of some) 'real Arnold complexity' and the (exponential of some) 'real topological entropy'. A detailed analysis of this 'real Arnold complexity' as a function of the parameter of this family of birational transformations of two variables is given. One can also slightly modify the definition of the dynamical zeta function, introducing a 'real dynamical zeta function' associated with the counting of the real cycles only. Similarly, one can also introduce some 'real Arnold complexity' generating functions. We show that several of these two 'real' generating functions seem to have the same singularities. Furthermore, we actually conjecture several simple rational expressions for them, yielding again algebraic values for the (exponential of the) 'real topological entropy'. In particular, when the parameter of our family of birational transformations becomes large, we obtain two interesting compatible nontrivial rational expressions. These rational results for real mappings cannot be understood in terms of any obvious Markov's partition, or symbolic dynamics hyperbolic systems interpretation: the birational transformation is far from being hyperbolic, it is measure preserving. It would be useful to know whether this relation between the Arnold complexity and the topological entropy, as well as the rationality of the degree generating functions and dynamical zeta functions, are a consequence of the measure-preserving property of the mapping.

1. Introduction

The purpose of this paper is to sketch a classification of birational transformations based on various notions of 'complexity'. In previous papers [1–3] an analysis, based on the examination

of the successive (bi)rational expressions corresponding to the iteration of some given birational mappings, has been performed. When one considers the degree $d(N)$ of the numerators (or denominators) of the corresponding successive rational expressions for the Nth iterate, the growth of this degree is (generically) exponential with N: $d(N) \simeq \lambda^N$. The quantity λ has been called the 'growth complexity' [4] and it is closely related to the *Arnold complexity* [5]. A semi-numerical analysis, enabling one to compute these growth complexities λ for these birational transformations, has been introduced in [1, 2]. It has been seen, on particular sets of birational transformations [6], that these 'growth complexities' correspond to a remarkable spectrum of *algebraic* values [4].

These 'growth complexities', summing up the (asymptotic) evolution of the *degree* of the successive iterates, amount to viewing these mappings as mappings of (two) *complex* variables. However, when one considers the phase portrait of these mappings, one also gets some 'hint' of the 'complexity' of these mappings seen as mappings of (two) *real* variables. In the following, we will consider a one-parameter-dependent birational mapping of two variables. On this very example, it will be seen, considering phase portraits corresponding to various values of the parameter, that these 'real complexities' vary for the different (positive) values of the parameter. Two universal (or 'topological') measures of the complexities were found to identify [1, 2], namely the (multiplicative rate of growth of the) *Arnold complexity* [5] (or growth complexity) and the (exponential of the) *topological entropy* [1]. The topological entropy, $\ln(h)$, is associated with the exponential growth h^N of the number of fixed points (real or complex) of the Nth iterate of the mapping: looking at various phase portraits, corresponding to different values of the parameter (see below), it is tempting to define, in an equivalent way, a 'real topological entropy' associated with the exponential growth h_{real}^N of the number of *real fixed points only* of the Nth iterate of the mapping. This notion of 'real topological entropy' would actually correspond to the 'visual complexity' as seen on the phase portrait of the mapping. Such a concept, corresponding to the evaluation of the real complexity h_{real} of the mapping seen as a mapping bearing on *real* variables, would be less universal: it would have only 'some' of the remarkable topological universal properties of the topological entropy. Similarly, it is also tempting to slightly modify the definition of the *Arnold complexity* [5]. The Arnold complexity [5], which corresponds (at least for the mappings of two variables) to the degree growth complexity [2, 3], is defined as the number of (real or complex) intersections of a given (generic and complex) line with its Nth iterate: it is straightforward to similarly define a notion of 'real Arnold complexity' describing the number of *real* intersections of a given (generic) *real* line with its Nth iterate. This real-analysis concept is, at first sight, also very well suited to describe the 'real complexity' of the mapping as seen in the phase portrait (see figure 2). Recalling the identification, seen on this one-parameter family of birational mappings, between the (multiplicative rate of growth of the) Arnold complexity and the (exponential of the) topological entropy [1, 2], it is natural to wonder if this identification also works for their 'real' partners, or if, as common sense would suggest, real analysis is 'far less universal', depending on a lot of details and, thus, requires a 'whole bunch' of 'complexities' (Lyapounov dimensions, . . .) to be described properly.

In order to see the previous identification even more clearly, one can also slightly modify the definition of the dynamical zeta function, introducing a 'real dynamical zeta function' associated with the counting of the real cycles only, and, similarly, one can also introduce some 'real Arnold complexity' generating functions. We will show that several of these two 'real' generating functions seem to have the same singularities. Furthermore, we will actually conjecture several *simple rational expressions* for them, yielding, again, *algebraic values* for the (exponential of the) 'real topological entropy'. In particular, when the parameter of our family of birational transformations becomes large, we will get an interesting nontrivial

rational expression. These rational results for real mappings cannot be simply understood by any 'obvious' Markov's partition, or symbolic dynamics hyperbolic interpretation.

Let us first recall, in the following two sections, some previous results† and notations.

2. Growth (Arnold) complexity for a birational mapping

A *one-parameter* family of birational mappings of *two* (complex) variables has been introduced in previous papers [2, 7, 8] (see definition (3) in [2]). This mapping actually originates from a lattice statistical mechanics framework that will not be detailed here [7, 9–11]. This one-parameter family of maps is a particularly interesting test family as it is integrable for a certain set of values of the parameter, has 'nongeneric' behaviour at a certain countable set of values of the parameter, and has 'generic' behaviour at all other values. Furthermore, these maps have quite complicated dynamics, yet the maps themselves are sufficiently simple to perform some explicit theoretical analysis. In the following, we will use the extreme simplicity of this mapping of *two* (complex) variables to first compare two quite universal (topological) notions of 'complexity' namely the growth complexity λ, which measures the exponential growth of the *degree* of the successive rational expressions encountered in an iteration (a notion which coincides with the (multiplicative rate of growth of the) *Arnold complexity*‡ [5]), and the (exponential of the) *topological entropy* [12, 13]. In section 4.3, we will go a step further and compare, more particularly, the notion of *'real' Arnold complexity* versus the notion of *'real' topological entropy*. These two notions will be seen to be suitable to describe the properties of the mapping seen as a mapping of *real* variables.

2.1. A one-parameter family of birational transformation

Let us consider the following birational transformation (see (3) in [2]) of two (*complex*) variables k_ϵ, depending on one parameter ϵ:

$$k_\epsilon : (y, z) \longrightarrow (y', z') = \left(z + 1 - \epsilon, \frac{z - \epsilon}{z + 1} \cdot y \right). \tag{1}$$

This map is the product of two involutions, $I_1 : (y, z) \longleftrightarrow (-z, -y)$ and

$$I_2 : (y, z) \longrightarrow (y', z') = \left(-\frac{z - \epsilon}{z + 1} \cdot y, \epsilon - 1 - z \right). \tag{2}$$

These two involutions have the lines $z = -y$ and $z = (\epsilon - 1)/2$ as fixed-point sets, respectively. The inverse transformation k_ϵ^{-1} is nothing but transformation (1) where $y \leftrightarrow -z$:

$$(y, z) \longrightarrow (y', z') = \left(\frac{y + \epsilon}{y - 1} \cdot z, \epsilon - 1 + y \right). \tag{3}$$

In spite of its simplicity, birational mapping (1) can, however, have quite different behaviours according to the actual values of the parameter ϵ. For example, for $\epsilon = 0$, as well as $\epsilon = -1, \frac{1}{2}, \frac{1}{3}$ or 1, the mapping becomes *integrable*, whereas it is not [8] for all other values of ϵ.

Let us now compare, in the following, two notions of 'complexity' (Arnold complexity versus topological entropy) according to various values of ϵ.

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† Note, however, that sections 2.3, 3.1, 3.3, and 3.4 provide new results.

‡ More precisely the Arnold complexity $C_A(N)$ is proportional (for plane maps) to $d(N)$, the degree of the Nth iteration of the birational mapping which behaves like $d(N) \simeq \lambda^N$. This 'degree notion' was also introduced by Veselov in exact correspondence with the general Arnold definition [5]. Note that the concept of Arnold complexity *is not* restricted to two-dimensional maps.
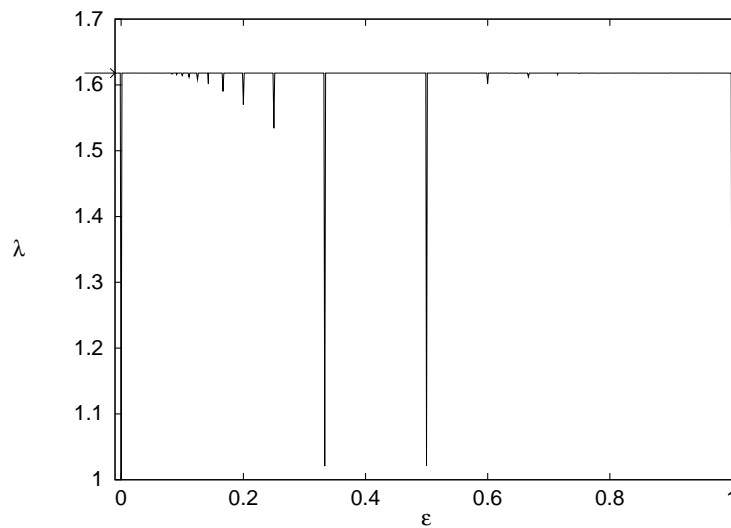
Figure 1. Complexity λ, for k_ϵ, as a function of ϵ.

2.2. Semi-numerical approach for the growth complexity λ

The (growth) complexity λ, which measures the exponential growth of the degrees of the successive rational expressions one encounters in the iteration of the birational transformation (1), can be obtained by evaluating the degrees of the numerators, or equivalently of the denominators, of the successive (bi)rational expressions obtained in the iteration process. One can actually build a *semi-numerical method* [1, 2] to get the value of the complexity growth λ for any† value of the parameter ϵ. The idea is to iterate, with the birational transformation (1), a generic *rational* initial point (y_0, z_0) and to follow the magnitude of the successive numerators, or denominators, of the iterates. During the first few steps some accidental simplifications may occur, but, after this transient regime, the integer denominators (for instance) grow like λ^n, where n is the number of iterations. Typically, a best fit of the logarithm of the numerator as a linear function of n, between $n = 10$ and 20, gives the value of λ within an accuracy of 0.1%. Let us remark that an *integrable mapping* yields a *polynomial growth* of the calculations [10]: the value of the complexity λ has to be numerically very close to 1.

Figure 1 shows the values of the complexity growth λ as a function of the parameter ϵ. One should note from figure 1 that all the values of ϵ (except a zero measure set) give a growth complexity $\lambda \simeq 1.618$. The calculations have been performed using an infinite-precision‡ C-library [14]. This semi-numerical analysis [2] clearly indicates that, beyond the known integrable values [8] of ϵ, namely $-1, 0, \frac{1}{3}, \frac{1}{2}, 1$, two sets of values $\{\frac{1}{4}, \frac{1}{5}, \frac{1}{6}, \ldots, \frac{1}{13}\}$ and $\{\frac{3}{5}, \frac{2}{3}, \frac{5}{7}\}$ are singled out. This suggests that the growth complexity λ takes lower values than the generic one on two infinite sequences of ϵ values, namely $\epsilon = 1/n$ and $\epsilon = (m-1)/(m+3)$ for n and m integers such that $n \geqslant 4$ and $m \geqslant 7$ and m odd.

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† Due to the use of high precision arithmetic in these computer calculations, the values of ϵ are only rational numbers.
‡ The multi-precision library gmp (GNU MP) is part of the GNU project. It is a library for *arbitrary* precision arithmetic, operating on signed integers, rational numbers and floating point numbers. It is designed to be as fast as possible, both for small and huge operands. The current version is 2.0.2. Targeted platforms and software/hardware requirements are any Unix machines, DOS and others, with an operating system with reasonable include files and a C compiler.

2.3. Generating functions for the degree growth of the successive iterates

Performing exact formal (Maple) calculations, one can revisit all these results using the stability of some factorization schemes [2,4,6] associated with these iteration calculations. For instance, one can consider, for various values of ϵ, the degrees of the successive rational expressions one encounters when performing the successive iterates, and build various generating functions† corresponding to these successive degrees. In particular, having singled out a set of ϵ values, one can revisit these various values, to see how the generic growth complexity $\lambda \simeq 1.618$ becomes modified, and deduce the degree generating functions, and the associated complexity λ, in each case. Let us denote by $G_\epsilon(t)$ the degree generating function, corresponding, for some given value of the parameter ϵ, to the degree of (for instance) the numerator of the z component of the successive rational expressions obtained in the iteration process of transformation (1).

At this stage, it is worth recalling, again, the notion of Arnold complexity [5] which corresponds to iterating a given (complex) line and counting the number A_N of intersections of this Nth iterate with the initial line. It is straightforward to see that these Arnold complexity numbers A_N are closely linked to these successive degrees (see, for instance, [1,2]). Actually, if one considers the iteration of the $y = (1 - \epsilon)/2$ line‡, the generating function of the A_N *'almost' identifies* (often up to a simple $t/(1 + t)$ factor) with the degree generating functions $G_\epsilon(t)$. The expansions of the 'Arnold' generating functions $A_\epsilon(t)$, and the degree generating functions $G_\epsilon(t)$, are in agreement (up to order 15 for the 'Arnold' generating functions and order 10, or 11, for the degree generating functions) with the following rational expressions (for $m \geqslant 4$):

$$A_\epsilon(t) = \frac{t}{1+t} \cdot G_\epsilon(t) = \frac{t}{1 - t - t^2}$$

$$A_{1/m}(t) = \frac{t}{1+t} \cdot G_{1/m}(t) = \frac{t}{1 - t - t^2 + t^{m+2}}$$

$$A_{(m-1)/(m+3)}(t) = \frac{t}{1+t} \cdot G_{(m-1)/(m+3)}(t) = \frac{t}{1 - t - t^2 + t^{m+2}} \tag{4}$$
$$\text{for} \quad m = 9, 13, 17, 21, \ldots$$

$$A_{(m-1)/(m+3)}(t) = \frac{t \cdot (1 - t^{(m+1)/2})}{1+t} \cdot G_{(m-1)/(m+3)}(t) = \frac{t \cdot (1 - t^{(m+1)/2})}{1 - t - t^2 + t^{m+2}}$$
$$\text{for} \quad m = 7, 11, 15, \ldots$$

where the expression for $G_\epsilon(t)$ is valid for ϵ generic and the expressions for $G_{1/m}(t)$ are valid for $m \geqslant 4$, the $G_{(m-1)/(m+3)}(t)$ for $m \geqslant 7$ with m odd. One also has for various integrable value of ϵ:

$$A_{-1}(t) = \frac{t}{1+t} \cdot G_{-1}(t) = \frac{t}{1 - t^2} \qquad A_0(t) = \frac{t}{1+t} \cdot G_0(t) = \frac{t}{(1-t)(1+t)}$$

$$A_1(t) = \frac{t \cdot (1 - t)}{1+t} \cdot G_1(t) = \frac{t}{1 - t^2}$$

$$A_{1/3}(t) = \frac{t \cdot (1+t) \cdot (1-t)^2}{1+t^4} \cdot G_{1/3}(t) = \frac{t \cdot (1+t)}{1 - t^3} \tag{5}$$

$$A_{1/2}(t) = \frac{t}{1+t} \cdot G_{1/2}(t) = \frac{t \cdot (1 - t^9)}{(1-t) \cdot (1-t^2) \cdot (1-t^3) \cdot (1-t^5)}.$$

† Similar calculations of generating functions have been performed [2] using other representations of the mapping related to 3×3 matrices [11]. These generating functions, denoted $G_\epsilon(x)$ in [2], are deduced from the existence of remarkable stable *factorization schemes* [2,4,6]. These results are in complete agreement with the one given here for the mapping of two variables (1) (see for instance equations (11)–(14) in [2]).

‡ Which is known to be a singled-out line for this very mapping [2] (see also, sections 4.2, 5). However, similar calculations can be performed with any other 'generic' line (which excludes, for instance, the line $y = z + 1$ which has no fixed points of k_ϵ^N for any N). We have found, in our computer experiments, that $y = (1 - \epsilon)/2$ (or $z = (\epsilon - 1)/2$) seem to yield more 'regular' complexity numbers A_N.

These various 'exact' generating functions are in agreement with the previous semi-numerical calculations. In particular, the first expression in (4) yields an algebraic value for λ in agreement with the generic value of the complexity $\lambda \simeq 1.618$ of figure 1 (and figure 1 in [2]).

Remark. Due to the use of infinite precision computer calculations, one may think that our analysis is only restricted to rational values of ϵ. This is not the case: using formal calculations one can also perform this analysis of the successive degrees, or successive A_N, for *arbitrary values* of ϵ. It can be shown that the singled-out values of ϵ for which λ could be different from the generic $\lambda \simeq 1.618$ value can only be *algebraic numbers* (like (40) and (41) sketched in section 4.2 including, of course, the previously mentioned $1/m$ and $(m-1)/(m+3)$ rational values). Of course, these formal calculations are much more 'time consuming' and can only be used to analyse in detail a few specific values of ϵ.

3. Dynamical zeta function and topological entropy

It is well known that the periodic orbits (cycles) of a mapping k strongly 'encode' dynamical systems [15]. The fixed points of the Nth power of the mapping being the cycles of the mapping itself, their proliferation with N provides a 'measure' of chaos [16, 17]. To keep track of this number of cycles, one can introduce the fixed-points generating function

$$H(t) = \sum_N \#\text{fix}(k^N) \cdot t^N \tag{6}$$

where $\#\text{fix}(k^N)$ is the number of fixed points of k^N, *real or complex*. This quantity only depends on the number of fixed points, and *not* on their particular localization. In this respect, $H(t)$ is a *topologically invariant quantity*. The same information can also be coded in the so-called† *dynamical zeta function* $\zeta(t)$ [13, 19] related to the generating function $H(t)$ by $H(t) = t \frac{\mathrm{d}}{\mathrm{d}t} \log(\zeta(t))$. The dynamical zeta function is defined as follows [15, 18, 19]:

$$\zeta(t) = \exp\left(\sum_{N=1}^{\infty} \#\text{fix}(k^N) \cdot \frac{t^N}{N} \right). \tag{7}$$

The topological entropy [12] $\log h$ is‡

$$\log h = \lim_{N \to \infty} \frac{\log(\#\text{fix}(k^N))}{N}. \tag{8}$$

If the dynamical zeta function is *rational*, h will be the inverse of the pole of smallest modulus of $H(t)$ or $\zeta(t)$. If the dynamical zeta function can be interpreted as the ratio of two characteristic polynomials of two linear operators† A and B, namely $\zeta(t) = \det(1 - t \cdot B)/\det(1 - t \cdot A)$, then the number of fixed points $\#\text{fix}(k^N)$ can be expressed from $\text{Tr}(A^N) - \text{Tr}(B^N)$. In this case, the poles of a rational dynamical zeta function are related to the (inverse of the zeroes of the) characteristic polynomial of the linear operator A only. Since the number of fixed points remains unchanged under *topological conjugacy* (see Smale [24] for this notion), the

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† The dynamical zeta function has been introduced by analogy with the Riemann ζ function, by Artin and Mazur [18].
‡ This definition (see for instance [13, 20]) is not the 'standard' definition mathematicians are used to, namely a topological entropy defined for a *continuous* map of a compact set. However, since we are not interested in *flows* but rather in *discrete maps* we prefer to take a definition for the topological entropy in terms of the rate of growth of periodic points.
† For more details on these Perron–Frobenius, or Ruelle–Araki transfer operators, and other shifts on Markov partition in a symbolic dynamics framework see, for instance, [19, 21–23]. In this linear operators framework, the *rationality* of the zeta function, and therefore the algebraicity of the (exponential of the) topological entropy, amounts to having a *finite-dimensional representation* of the linear operators A and B.

dynamical zeta function is also a *topologically invariant function*, invariant under a large set of transformations, and does not depend on a specific choice of variables. Such invariances were also noticed for the growth complexity λ. It is thus tempting to make a connection between the *rationality* of the complexity generating function previously given, and a possible *rationality* of the dynamical zeta function. We will also compare the singularities of these two sets of generating functions, namely the growth complexity λ, and h the (exponential of the) topological entropy.

Some results for the dynamical zeta function. Let us now obtain the expansion of the dynamical zeta function of the mapping k_ϵ, for generic values of ϵ. We can first concentrate on the specific‡, but arbitrary, value $\epsilon = \frac{21}{25}$. Of course, there is nothing special about this specific $\epsilon = \frac{21}{25}$ value: the same calculations have been performed for many other *generic* values of ϵ yielding the same number of (complex) fixed points and, thus, the same dynamical zeta function. The total number of fixed points of k_ϵ^N, for N running from 1 to 14, yields the following expansion for the generating function $H(t)$ of the number of fixed points:

$$H_\epsilon(t) = H_{21/25}(t) = t + t^2 + 4t^3 + 5t^4 + 11t^5 + 16t^6 + 29t^7 + 45t^8$$
$$+76t^9 + 121t^{10} + 199t^{11} + 320t^{12} + 521t^{13} + 841t^{14} + \cdots. \tag{9}$$

This expansion coincides with the one of the *rational* function§

$$H_\epsilon(t) = \frac{t \cdot (1 + t^2)}{(1 - t^2) \cdot (1 - t - t^2)} \tag{10}$$

which corresponds to a very *simple‖ (conjectured) rational* expression for the dynamical zeta function

$$\zeta_\epsilon(t) = \frac{1 - t^2}{1 - t - t^2}. \tag{11}$$

An alternative way of writing the dynamical zeta functions relies on the decomposition of the fixed points into *irreducible cycles*:

$$\zeta_\epsilon(t) = \frac{1}{(1-t)^{N_1}} \cdot \frac{1}{(1-t^2)^{N_2}} \cdot \frac{1}{(1-t^3)^{N_3}} \cdots \frac{1}{(1-t^r)^{N_r}} \cdots. \tag{12}$$

For generic values of ϵ, one gets the following numbers of irreducible cycles: $N_1 = 1$, $N_2 = 0$, $N_3 = 1$, $N_4 = 1$, $N_5 = 2$, $N_6 = 2$, $N_7 = 4$, $N_8 = 5$, $N_9 = 8$, $N_{10} = 11$, $N_{11} = 18$, ... It has been conjectured in [2] that *the simple rational expression* (11) *is the actual expression of the dynamical zeta function for any generic value of ϵ* (up to some algebraic values of ϵ, see below and in section 4.2). Similar calculations have been performed for the other values of ϵ that have been singled out in the semi-numerical analysis [3]. For the nongeneric values of ϵ, $\epsilon = 1/m$ with $m \geqslant 4$, we have obtained expansions compatible with the following *rational* expression:

$$\zeta_{1/m}(t) = \frac{1 - t^2}{1 - t - t^2 + t^{m+2}}. \tag{13}$$

‡ Another generic value of ϵ, close to the $\frac{1}{2}$ value where the mapping is integrable [8], namely $\epsilon = \frac{13}{25} = 0.52$, has been analysed in some detail in [2]. For this value $\epsilon = 0.52$, the enumeration of the number of fixed points, n-cycles and the actual status of these fixed points (elliptic, hyperbolic, points . . .) are given in [2].

§ Valid for generic values of ϵ, up to some algebraic values of ϵ corresponding to cycle-fusion mechanisms—see (40) and (41) below and [3].

‖ As far as *symbolic dynamics* is concerned, one can associate, to a dynamical zeta function such as (11), a clipped Bernoulli shift with the 'pruning rule' to forbid substring $-11-$ (that is 1 must be always followed by 0) in any sequence of 0 and 1. However, constructing the Markovian partitions (if any!), yielding this simple pruning rule for the symbolic dynamics, has not been done: mapping (1) is *not hyperbolic* (even weakly hyperbolic), it will be seen below (see section 3.3) that it is *measure preserving*.

For the other nongeneric values, namely $\epsilon = (m-1)/(m+3)$ with $m \geqslant 7$ odd, the expansions are not large enough to conjecture a unique formula valid for any m. For $m = 7$ (namely $\epsilon = \frac{3}{5}$), one actually gets a dynamical zeta function given by (13) for $m = 7$ and this might also be the case for $m = 11, 15, \ldots$. For $m = 9, 13, \ldots$, the expansions are in agreement with a $(1 - t - t^2 + t^{m+2})$-singularity. Comparing the various rational expressions in (4) corresponding to generic, and nongeneric, values of ϵ, with (11) and (13), respectively, one sees that the singularities (poles) of the dynamical zeta function happen to *coincide* with the poles of the generating functions of the growth complexity λ, for all the values of ϵ. In particular, the growth complexity λ and h, the exponential of the topological entropy, are always *equal* for this very mapping.

Let us just mention, here, that the modification of the number of fixed points, from the 'generic' values of ϵ to the particular values $(1/m, (m-1)/(m+3))$, corresponds to the *disappearance of some cycles* which become singular points (indetermination of the form $0/0$). These mechanisms will be detailed in [25, 26]. Actually, the 'nongeneric' values of ϵ, like $\epsilon = 1/m$, correspond to such a 'disappearance of cycles' mechanism which modifies the denominator of the rational generating functions, and, thus, the topological entropy and the growth complexity λ. In contrast, there actually exists for k_ϵ, *other singled-out values* of ϵ, like $\epsilon = 3$ for instance, which correspond to *fusion of cycles* (see section 4.2): in the $\epsilon \to 3$ limit, the order-three cycle tends to coincide with the order-one cycle, which amounts to multiplying the dynamical zeta function (11) by $1 - t^3$. Such a 'fusion-cycle' mechanism *does not modify* the denominator of the rational functions, and thus the topological entropy and the growth complexity λ, *remain unchanged*.

To sum up.   Considering a (very simple) one-parameter-dependent birational mapping of *only two (complex) variables*, we have deduced an exact identification between the (multiplicative rate of growth of the) Arnold complexity, that is the growth complexity λ, and the (exponential of the) topological entropy for *all* the various ϵ cases (generic or not). As a byproduct, one finds that these two complexities correspond, in this very example, to *simple algebraic numbers*.

3.1. A canonical degree generating function

This identification result is not completely surprising: the dynamical zeta function is a quite 'universal' function, invariant under a large set of *topological conjugations* [24], and the concept of Arnold complexity (or the degree growth complexity λ) also has the same 'large' set of (topological and projective) invariances [5].

Actually, as far as degree generating functions are concerned, it is natural to introduce, instead of some generating functions of the degrees of the numerator of the z component of the Nth iterate, a more 'canonical' degree generating function $G_\epsilon^{\mathrm{Hom}}(t)$ associated with the birational mapping (1) written in a *homogeneous way* (see the bi-polynomial mapping (4) in [2]). Iterating (1), written in a homogeneous way, and, factoring out at each iteration step the greatest common divisors, one gets a new degree generating function $G_\epsilon^{\mathrm{Hom}}(t)$ well suited, at first sight, to describe such large (topological and projective) invariances.

A simple calculation shows that this (projectively well-suited) degree generating function reads (for generic ϵ)

$$G_\epsilon^{\mathrm{Hom}}(t) = \frac{1}{(1-t) \cdot (1 - t - t^2)}. \tag{14}$$

For the $\epsilon = 1/m$ particular values ($m \geqslant 4$), and for the two integrable values, $\epsilon = \frac{1}{2}$ and $\frac{1}{3}$,

one gets, respectively

$$G_{1/m}^{\text{Hom}}(t) = \frac{1 - t^{m+3}}{(1-t) \cdot (1-t-t^2+t^{m+2})} \tag{15}$$

$$G_{1/2}^{\text{Hom}}(t) = \frac{1-t^9}{(1-t)^2 \cdot (1-t^3) \cdot (1-t^5)} + \frac{t^2 \cdot (1-t^6)}{(1-t)^2 \cdot (1-t^2) \cdot (1-t^5)} \tag{16}$$

$$G_{1/3}^{\text{Hom}}(t) = \frac{1-t^6}{(1-t^3)(1-t^2)(1-t)^2} + \frac{t^4}{(1-t^3)(1-t)^2}. \tag{17}$$

Since the expansions for the infinite set of values of the form $(m-1)/(m+3)$ for $m \geqslant 7$, can only be performed up to order 11 (or 12), it is difficult to 'guess' any expression valid for any m such as (15). Recalling the results (4) given in section 2.3 for the degree growth generating functions, one may suspect that, among these $(m-1)/(m+3)$ for $m \geqslant 7$ values (m odd), one should make a distinction between $m = 7, 11, 15, \ldots$ on one side, and $m = 9, 13, 17, \ldots$ on the other side. In fact, up to order 11, all our calculations for various $(m-1)/(m+3)$ values for $m \geqslant 7$ ($\epsilon = \frac{3}{5}, \frac{2}{3}, \frac{5}{7}, \frac{3}{4}, \frac{7}{9}, \frac{4}{5}, \ldots$) are in agreement with a general equality between $G_{(m-1)/(m+3)}^{\text{Hom}}(t)$ and $G_{1/m}^{\text{Hom}}(t)$. More details are available in appendix A.

One gets simpler expressions for the integrable values $\epsilon = 0, 1$ and $\epsilon = -1$:

$$G_0^{\text{Hom}}(t) = G_1^{\text{Hom}}(t) = \frac{1+t^2}{(1-t)^2} \qquad \text{and} \qquad G_{-1}^{\text{Hom}}(t) = \frac{1}{1-t}. \tag{18}$$

Note that $G_\epsilon^{\text{Hom}}(t)$ verifies the simple functional equation $G_\epsilon^{\text{Hom}}(t) + G_\epsilon^{\text{Hom}}(1/t) = 1$, for $\epsilon = -1, \frac{1}{2}, \frac{1}{3}$, and $G_\epsilon^{\text{Hom}}(t) = G_\epsilon^{\text{Hom}}(1/t)$ for $\epsilon = 0, +1$.

For $\epsilon = \frac{1}{2}$ we have not written any dynamical zeta function $\zeta_{1/2}(t)$ since, for such an integrable birational mapping, there exists, at (almost) any order N of iteration, an *infinite number* of fixed points of order N (all the points of some elliptic curves [8]) and, therefore, our previous 'simple' definition (7) for the dynamical zeta function is no longer valid.

A possible universal relation. One can imagine many simple relations between the 'canonical' degree generating function, $G_\epsilon^{\text{Hom}}(t)$, and the dynamical zeta function, $\zeta_\epsilon(t)$. For instance, for a generic ϵ, one gets (among many . . .) the relation $(1-t) \cdot (1-t^2) \cdot G_\epsilon^{\text{Hom}}(t) = \zeta_\epsilon(t)$, however this relation is no longer valid for $\epsilon = 1/m$. One would like to find a 'true universal' relation between $\zeta_\epsilon(t)$ and $G_\epsilon^{\text{Hom}}(t)$, that is a relation *independent* of ϵ (generic or nongeneric). In order to achieve this goal one might think of exchanging $\zeta_\epsilon(t)$, and $G_\epsilon^{\text{Hom}}(t)$, for *projectively well-suited generating functions* taking into account the point at ∞, namely a dynamical zeta function taking into account the fixed point at ∞ (see (52) in [3]), $\zeta_\epsilon^{(\infty)}(t)$, and $G_\infty^{\text{Hom}}(\epsilon, t)$ defined as follows:

$$\zeta_\epsilon^{(\infty)}(t) = \frac{\zeta_\epsilon(t)}{1-t} \qquad \text{and} \qquad G_\infty^{\text{Hom}}(\epsilon, t) = G_\epsilon^{\text{Hom}}(t) + \frac{t}{1-t}. \tag{19}$$

One verifies immediately that the relation

$$G_\infty^{\text{Hom}}(\epsilon, t) = (1+t) \cdot \zeta_\epsilon^{(\infty)}(t) \qquad \text{or equivalently} \quad (1+t) \cdot \zeta_\epsilon(t) = (1-t) \cdot G_\epsilon^{\text{Hom}}(t) + t \tag{20}$$

is *actually verified for generic* values of ϵ, as well as for the *nongeneric values* of the form $\epsilon = 1/m$, and also some *nongeneric values* of the form $\epsilon = (m-1)/(m+3)$ (see appendix A). A similar relation for the two-parameters family of birational transformations depicted in [1–3] will be detailed elsewhere. Relation (20) should give some hint for a true mathematical proof of the relation between Arnold complexity and topological entropy (with some well-suited mathematical assumptions, see section 3.4).

Remark. Recalling the 'Arnold' generating functions $A_\epsilon(t)$, (see (4)), which identifies 'most of the time' (namely ϵ generic, $\epsilon = 1/m$, $m \geqslant 4$, $\epsilon = (m+1)/(m+3)$ for $m = 9, 13, 17, \ldots$) with the new well-suited generating function $G_\infty^{\mathrm{Hom}}(\epsilon, t)$, up to a simple multiplicative factor $t/(1+t)^2$, one can rewrite, for these values of ϵ (generic or not), relation (20) as

$$t \cdot \zeta_\epsilon(t) = (1 - t^2) \cdot A_\epsilon(t). \tag{21}$$

3.2. Status of these rational results

The expansions obtained here (and in the following) are not numerical results but *exact* results. Of course, given a power series up to order 14, there are many rational expressions (Padé approximants ...) which reproduce the power series up to this order. As far as the Arnold complexity generating functions are concerned, one must say that the occurrence of rational expressions for the (degree) Arnold complexity generating functions is a direct consequence of the *stability of some factorization scheme* [4, 6], describing the simplifications occurring when one iterates a birational transformations which originates from the composition of a matrix inverse and some permutations of the matrix entries [4, 6]. Of course one can imagine, for a very large number of iterations, some breaking of this factorization scheme. However, the very large number of birational examples we have studied [4, 6], strongly support the assumption of the stability of this factorization scheme, at least for this particular class of birational transformations. As far as the dynamical zeta function is concerned, one should not see (10) or (11) (and similar rational dynamical zeta functions given in the following ...) as the simplest possible Padé approximants among an infinite set of other possible rational expressions (with the same simplicity prejudice that is used to justify Padé approximant analysis in lattice statistical mechanics for instance), but rather as the relation

$$d_A(t) \cdot \zeta_\epsilon(t) = 1 - t^2 + \mathrm{O}(t^{15}) \tag{22}$$

where $d_A(t)$ is the denominator of the rational expression of the Arnold complexity generating function, namely $1 - t - t^2$.

In the framework of hyperbolic systems, or more generally weakly hyperbolic systems, or, in the even more general framework introduced by Conley [27] and extensively studied by Easton [29] or Fried [28], of *isolated blocks*, one may have a rationality prejudice for dynamical zeta functions. However, our rational results for (complex) birational mappings cannot be simply understood in term of any 'obvious' Markov's partition, or any symbolic dynamics hyperbolic interpretation: our mapping does not belong to the previous frameworks. Let us show here that our mapping is actually a *measure-preserving*† *mapping* [30].

3.3. Measure-preserving mapping

A *measure-preserving* mapping is a mapping that is conjugate to an *area-preserving* mapping: it can be rewritten, up to a quite complicated, and possibly singular transformation, into an *area-preserving* map [30]. Measure-preserving mappings were studied by Poincaré [31].

Calculating the Jacobian of transformation (1), one gets

$$\det \begin{bmatrix} \frac{\mathrm{d}y'}{\mathrm{d}y} & \frac{\mathrm{d}z'}{\mathrm{d}y} \\ \frac{\mathrm{d}y'}{\mathrm{d}z} & \frac{\mathrm{d}z'}{\mathrm{d}z} \end{bmatrix} = \det \begin{bmatrix} 0 & \frac{z-\epsilon}{z+1} \\ 1 & \frac{y(1+\epsilon)}{(z+1)^2} \end{bmatrix} = -\frac{z-\epsilon}{z+1}. \tag{23}$$

Let us note that the line $y = z + 1$ is a line where the successive points of the iterations 'seem' to accumulate (see the various figures in section 4.1). As far as seeking an invariant

† JMM would like to deeply thank R Quispel for an illuminating discussion on measure-preserving maps.

measure for mapping (1) is concerned, one should thus have a higher density near this singled-out line. The line $y = z + 1$ is actually a *globally* invariant line on which transformation (1) reduces to a *simple translation*:

$$k_\epsilon : (y, z) \longrightarrow (y - \epsilon, z - \epsilon). \tag{24}$$

Clearly *no fixed points of any order* can exist on this singled-out line. For generic values of ϵ this corresponds to the *only* (algebraic) covariant of transformation (1), namely $c(y, z) = y - 1 - z$ (of course, they are many more for integrable values of ϵ). Under transformation (1) the covariant $c(y, z) = y - 1 - z$ transforms with a cofactor which is simply the Jacobian (23):

$$k_\epsilon : y - 1 - z \longrightarrow y' - 1 - z' = -\frac{z - \epsilon}{z + 1} \cdot (y - 1 - z). \tag{25}$$

In other words, the Jacobian can always be written as the ratio $c(y', z')/c(y, z)$ (where $c(y', z')$ is the covariant taken at the image point (y', z')) which is the key ingredient for having a *measure-preserving* map (see relation (2.20) in [30]). Actually, introducing a change of variables $(y, z) \rightarrow (u, v)$ such that the Jacobian of this change of variables will be equal to the inverse of this covariant, will change our measure-preserving map into an *area-preserving map*:

$$\det \begin{bmatrix} \frac{du}{dy} & \frac{dv}{dy} \\ \frac{du}{dz} & \frac{dv}{dz} \end{bmatrix} = \frac{1}{y - 1 - z}. \tag{26}$$

They are an infinite number of such change of variables. One (not very elegant) solution amounts to imposing $(u, v) = (y, v(y, z))$, the Jacobian (26) reading

$$\frac{dv}{dy} = \frac{1}{y - 1 - z} \tag{27}$$

which can easily be integrated into $v = \ln(y - 1 - z)$, its inverse being $(y, z) = (u, u - e^v - 1)$. Rewriting the mapping in these (u, v) variables one gets

$$(u, v) \longrightarrow (u', v') = (u - e^v - \epsilon, v + \ln(V)) \qquad \text{where} \quad V = \frac{u - e^v - 1 - \epsilon}{e^v - u}. \tag{28}$$

One easily verifies that mapping (28) has a Jacobian equal to 1 everywhere. It is an *area-preserving* map.

As far as the previously introduced topological notions are concerned (dynamical zeta functions, Arnold complexity, topological entropy), it is clear that they remain unchanged for the area-preserving mapping (28).

To our knowledge, there is no nontrivial† result available in the literature on the possible rationality of the dynamical zeta function of measure-preserving, or area-preserving mappings. To our knowledge, there is no nontrivial result available on the possible relation (identification) between Arnold complexity and topological entropy for area-preserving maps.

3.4. Some comments on the relations between various entropies and complexity measures

One should recall that relations between degree complexity and topological entropy are extensively discussed in the mathematical literature (see e.g. Katok and Hasselblatt [32]). We do not want to mention here the well known inequalities between the metric entropy and the topological entropy, or even the more general order-q Renyi entropies [33]: this will be the subject of a forthcoming publication. We want, here, to look at the relations between two

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† Of course, one can certainly find many area-preserving mappings for which h, the exponential of the topological entropy, (or λ) is an *integer*, namely the degree of the associated homogeneous polynomial transformation. We are interested in nontrivial mappings for which h, or λ, are algebraic numbers that are *not integers*.

topological complexities, namely the topological entropy and Arnold complexity, and, in the following, their real adaptations. In this respect, one must certainly mention the relations, and inequalities, given by Newhouse [34] relating the topological entropy of a smooth map to the growth rates of the volumes of iterates of smooth manifolds†. For C^∞-smooth mappings, Yomdin proved the opposite inequality, thus showing the coincidence of the growth rate of volumes and topological entropy [36, 37]. One should also recall the paper by Friedland [38] which shows that the entropy is the same as the volume growth for rational self-maps of complex projective space P^2: in that case, the Arnold complexity coincides with the growth of homology [39] which should be the same as the volume growth.

In fact, it is not completely clear whether one can use all these mathematical theorems for our birational measure-preserving mappings. When mathematicians study birational transformations they tend to focus on the indeterminacy set where a birational map cannot be defined and are very worried about the bad things that might arise when this set grows with the iteration. In order to avoid such mathematically unpleasant proliferation of singularities they work in a framework which is a very 'smooth' one, with a point of departure of diffeomorphisms. The conceptual framework, and even the definitions of the topological entropy, being slightly different, it is difficult to see if these theorems really apply. Let us just point out here that, fortunately, the indeterminacy locus is far from being a dense set: it is very tame for mappings (1). From Jacobian (23), one gets that the critical locus is the line $z = \epsilon$, its critical image being the point $(y, z) = (1, 0)$. This is a point of indeterminacy for k_ϵ^{-1}. By inspection (or from the $y \leftrightarrow -z$ symmetry) the point of indeterminacy for k_ϵ is $(y, z) = (0, -1)$. Both critical points belong to the singled-out line $y = 1 + z$ on which the action of k_ϵ (or k_ϵ^{-1}) reduces to a simple shift (see (24)). The backwards and forwards iterates of these two critical points can thus be easily described‡.

From a more 'down-to-earth' point of view the comparison between topological entropy and Arnold complexity can be understood as follows. The components of $k_\epsilon^N(y, z)$, namely y_N and z_N, are of the form $P_N(y, z)/Q_N(y, z)$ and $R_N(y, z)/S_N(y, z)$, where $P_N(y, z)$, $Q_N(y, z)$, $R_N(y, z)$ and $S_N(y, z)$ are polynomials of degree asymptotically growing like λ^N. The Arnold complexity amounts to taking the intersection of the Nth iterate of a line (for instance, a simple line like $y = y_0$, where y_0 is a constant) with another simple (fixed) line (for instance $y = y_0$ itself or any other simple line or any *fixed* algebraic curve). Let us consider the Nth iterate of the $y = y_0$ line, which can be parametrized as

$$y_N = \frac{P_N(y_0, z)}{Q_N(y_0, z)} \qquad z_N = \frac{R_N(y_0, z)}{S_N(y_0, z)} \tag{29}$$

with line $y = y_0$ itself. The number of intersections, which are the solutions of $P_N(y_0, z)/Q_N(y_0, z) = y_0$, grows like the degree of $P_N(y_0, z) - Q_N(y_0, z) \cdot y_0$: asymptotically it grows like $\simeq \lambda^N$. On the other hand, the calculation of the topological entropy corresponds to the evaluation of the number of fixed points of k^N, that is, the number of intersections of the *two* curves: $P_N(y, z) - Q_N(y, z) \cdot y = 0$ and $R_N(y, z) - S_N(y, z) \cdot z = 0$ which are two curves of degree growing asymptotically like $\simeq \lambda^N$. The number of fixed points is obviously bounded by $\simeq \lambda^{2N}$. The exponential of the topological entropy, namely h, is thus bounded by the square of λ: $h \leqslant \lambda^2$.

In fact, we have found a possible example where this upper bound seems to be actually reached. Let us consider the quadratic transformation ($B \cdot C \neq 1$):

$$(y, z) \longrightarrow ((A - y - Bz) \cdot y \qquad (A - Cy - z) \cdot z). \tag{30}$$

† Schub conjectured that the topological entropy of a smooth map on a compact manifold is bounded by the growth of the various algebraic transformations that it induces [35].
‡ Furthermore, this also enables us to understand why the values $\epsilon = 1/m$, $m = 1, 2, 3, 4 \ldots$ are special.

The mapping is not bipolynomial or birational. The dynamical zeta function reads (up to order 5 only, the calculations becoming really large) for this noninvertible mapping:

$$\zeta(t) = \frac{1}{(1-t)^4(1-t^2)^5(1-t^3)^{20}(1-t^4)^{60}(1-t^5)^{204}} \cdots \tag{31}$$

from which one can conjecture that

$$\zeta(t) = \frac{1-t^2}{1-4t}. \tag{32}$$

This provides an example for which $h = \lambda^2 = 4$. Therefore, it seems that the identification of h and λ is not valid in general†. It seems that the identification of h and λ might be related to the 'very tame' *proliferation of singularities* of the (birational) transformations (1), or it might be a consequence of the measure-preserving property of the mapping. This is a quite complicated analysis that we do not want to sketch here. Let us just say that this identification seems to be a valid one in our particular example (1).

4. Real dynamical zeta function and real topological entropy

As far as the growth complexity λ is concerned, the generic values of ϵ (that is the values different from the previous $1/m$, $(m-1)/(m+3)$ singled-out values) are all on the same 'complexity footing' (see figure 1). This is clearly confirmed by the exponential *growth of the computing time* during the iteration process, which seems to be similar for all these values (and clearly *smaller* for the $1/m$, $(m-1)/(m+3)$ particular values). It is, however, worth noting that these generic values, which are all on the same λ-footing, clearly yield phase portraits which are quite different and, obviously, correspond to *drastically different* 'visual complexities' of the phase portrait of the mapping. This 'visual complexity' corresponds to the (exponential) growth of the number of (real) fixed points of the mapping seen as a mapping *bearing on two real variables*. The previous definitions of the dynamical zeta function $\zeta_\epsilon(t)$ and of the generating function $H_\epsilon(t)$ counting the number of fixed points, can be straightforwardly modified to describe the counting of *real fixed points*:

$$H_{\text{real}}(t) = \sum_N H_N^R \cdot t^N = t \cdot \frac{\mathrm{d}}{\mathrm{d}t}\log(\zeta^{\text{real}}(t)) \qquad \text{where} \quad \zeta^{\text{real}}(t) = \sum_N z_N^R \cdot t^N \tag{33}$$

where the number of *real* fixed points H_N^R grow exponentially with the number N of iterates, like $\simeq h_{\text{real}}^N$. A quick examination of various phase portraits for various 'generic values' of the parameter ϵ seems to indicate quite clearly that this 'real topological entropy' $\log(h_{\text{real}})$ *varies with ϵ*, in contrast with the 'usual' topological entropy $\log(h)$. An obvious inequality is $h_{\text{real}} \leqslant h$.

4.1. 'Phase portrait gallery'

Let us give here various phase portraits corresponding to different (generic except the first one) values of ϵ. Note the different scales for the frames of these various phase portraits. For most of the phase portraits given in figures 2–9 around 300 orbits of length 1000, starting from randomly chosen points inside the frame†, have been generated (only points inside the frame are shown).

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† One can imagine that some equality like $h = \lambda \cdot h_{\text{sing}}$ could be valid, where h_{sing} could correspond to the exponential proliferation of bifurcations or singularities. Such speculative ideas remain to be studied.
† With a special nonrandom treatment of the regular elliptic parts of the phase portraits.
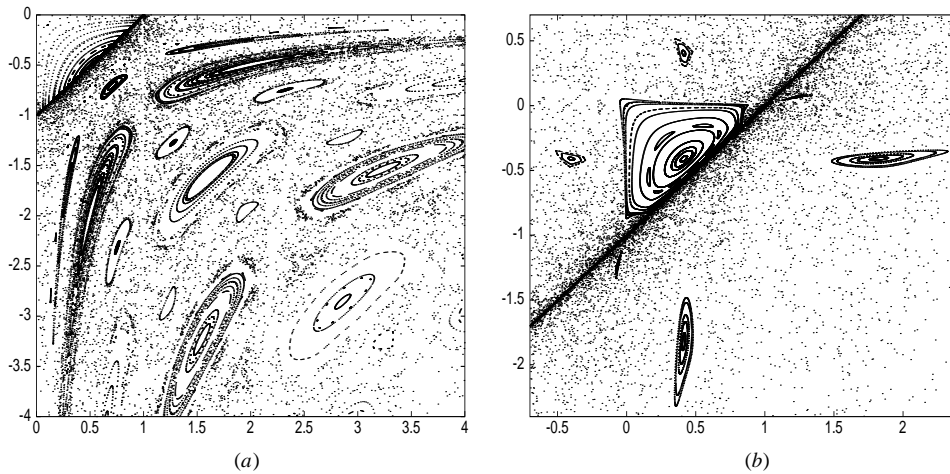
Figure 2. Phase portraits of k_ϵ for $\epsilon = \frac{1}{100}$ (left) and for $\epsilon = \frac{9}{50}$ (right).
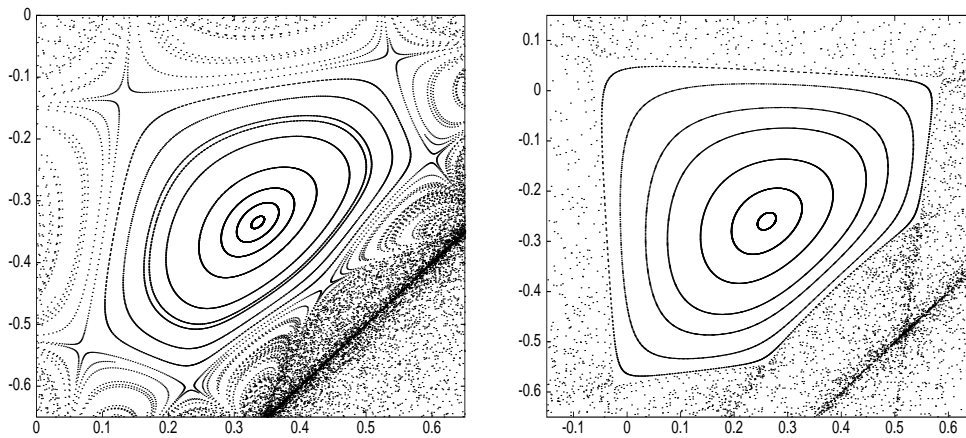

Figure 3. Phase portrait of k_ϵ for $\epsilon = \frac{33}{100}$.

Figure 4. Phase portrait of k_ϵ for $\epsilon = \frac{48}{100}$.

On these various phase portraits, one gets, near the integrable value $\epsilon \simeq \frac{1}{3}$, quite different phase portraits which seem, however, to have roughly the same number of (real) fixed points (see figures 3 and 6). On these various phase portraits, one also sees, quite clearly, that the number of fixed points seems to *decrease* when ϵ crosses the integrable value $\epsilon = \frac{1}{2}$ and $\epsilon = 1$, going, for instance, from $\epsilon = 0.48$ to 0.51 (see figures 4 and 7) or from $\epsilon = 0.9$ to 1.1 (see figures 5 and 8). These results will be revisited in section 5. Of course, exactly on the integrable value $\epsilon = 1$, the phase diagram corresponds to a (simple) foliation of the two-dimensional parameter space in *(rational) curves* (linear pencil of rational curves, see [8]):

$$\Delta(y, z, 1) = \left(\frac{yz}{y - z - 1} \right)^2 = \rho \qquad \text{or equivalently} \quad \frac{yz}{y - z - 1} = \pm\rho^{1/2} \tag{34}$$

where ρ denotes some constant. For the other integrable values one also has either a linear pencil of rational curves, namely $yz = \rho$ for $\epsilon = 0$, as well as

$$\Delta(y, z, -1) = \frac{1}{(1 + z - y)^2} = \rho \qquad \text{or equivalently} \quad (y - z) \cdot (y - z - 2) = \frac{1}{\rho} - 1 \tag{35}$$
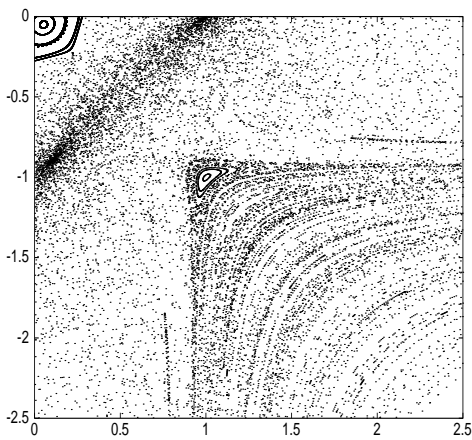
Figure 5. Phase portrait of k_ϵ for $\epsilon = \frac{9}{10}$.
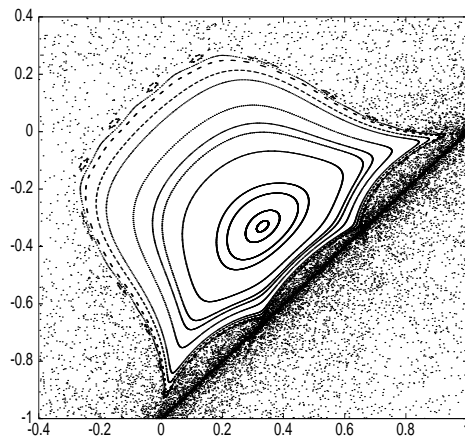

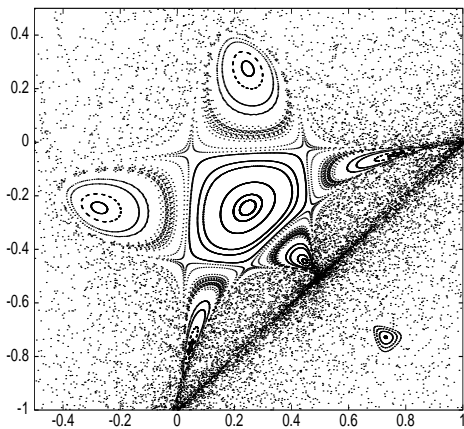**Figure 6.** Phase portrait of k_ϵ for $\epsilon = \frac{34}{100}$.

Figure 7. Phase portrait of k_ϵ for $\epsilon = \frac{51}{100}$.
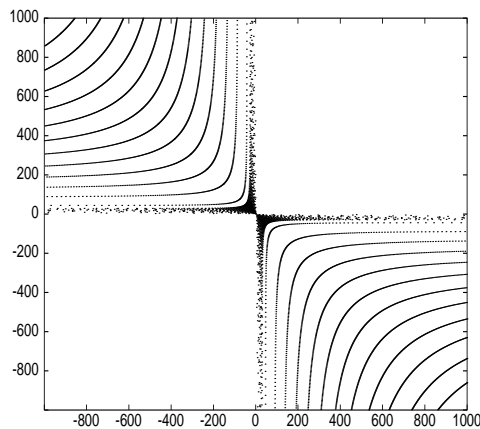
Figure 8. Phase portrait of k_ϵ for $\epsilon = \frac{11}{10}$.

for $\epsilon = -1$, or a linear pencil of *elliptic curves* for $\epsilon = \frac{1}{2}$, namely

$$\Delta(y, z, \tfrac{1}{2}) = \frac{(1 + z + 2yz) \cdot (1 - y + 2yz) \cdot (1 + z - y - 2yz)}{(1 + z - y)^2} = \rho \qquad (36)$$

and also

$$\Delta(y, z, \tfrac{1}{3}) = ((5 + 3z - 3y + 9yz) \cdot (1 - z - y + 3yz) \cdot (1 + z - y - 3yz)$$
$$\times (1 + z + y + 3yz))/(1 + z - y)^2 = \rho$$

for $\epsilon = \frac{1}{3}$. One also remarks that $\epsilon = 3$, which corresponds to the generic $\lambda \simeq 1.618$ growth complexity, also yields a *remarkably regular* phase portrait, 'visually' similar to a foliation of the two-dimensional parameter space in curves, suggesting a 'real topological complexity' h_{real} very close, or even equal to 1. This fact will also be revisited in section 5. In order to describe, less qualitatively, the 'real topological complexity' h_{real} as a function of the parameter ϵ, we have calculated in section 4.3, the first (10, 11 or even 12) coefficients of the expansions of $H_\epsilon^{\text{real}}(t)$, and of the 'real dynamical function' $\zeta_\epsilon^{\text{real}}(t)$, for various values of ϵ.

Figure 9. Phase portraits of k_ϵ for $\epsilon = 3$ (left) and for $\epsilon = 15$ (right).

4.2. Number of real fixed points as a function of ϵ

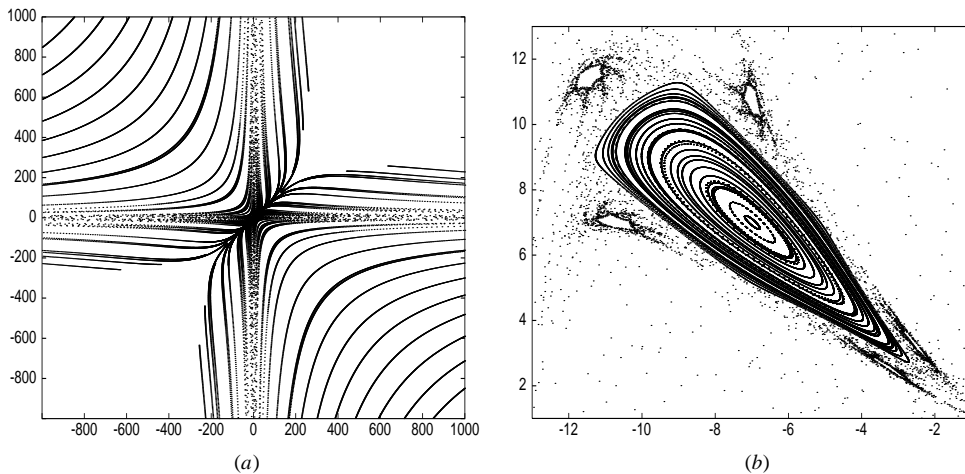Let us now try to understand why (and how) h_{real} varies as a function of ϵ, and why some other values of ϵ, like $\epsilon = 3$, different from the previous $1/m$ and $(m-1)/(m+3)$ nongeneric values, seem to play a special role. The method used to obtain the fixed points of the Nth iterate of k_ϵ has been detailed in previous papers [1,2]. Let us just mention here that, due to the symmetries of this mapping, there exist four singled-out lines, namely $y = (1 - \epsilon)/2$, $z = (\epsilon - 1)/2$, $y = -z$ and $y = z + 1$. The inverse transformation k_ϵ^{-1} being identical to k_ϵ where y has been permuted with $-z$, one can understand the line $y = -z$ as a 'mirror' between past and future in the iteration process. From the decomposition (2) of (1) into involutions I_1 and I_2 one can easily show that if $\mathcal{M} = (y, z)$ is a fixed point of order N of (1), its image $I_1(\mathcal{M}) = (-z, -y)$ is also a fixed point of order N. Recalling the Jacobian of transformations k_ϵ and k_ϵ^{-1}, one easily verifies that these Jacobians (see (23)) are equal to 1 on the lines $y = (1 - \epsilon)/2$ and $z = (\epsilon - 1)/2$, respectively. Line $z = (\epsilon - 1)/2$ is also the set of fixed points of involution (2).

An analysis of the set of the first-order fixed points ($N \leqslant 8$) shows that the first three lines, namely $y = (1-\epsilon)/2$, $z = (\epsilon - 1)/2$ and $y = -z$, play a key role in classifying all these fixed points [40]. It can be seen, for $N \leqslant 8$, that there always exist a fixed point in these N-cycles which belongs, either to line $y = -z$ or to the $y = (1 - \epsilon)/2$ (or equivalently $z = (\epsilon - 1)/2$) line. We call the fixed points, corresponding to line $y = -z$, the 'P-type' points. We call the fixed points, having a representative on $y = (1 - \epsilon)/2$, or equivalently $z = (\epsilon - 1)/2$, the 'Q-type' points [25, 26].

For $N \geqslant 9$, other N-cycles with no points lying on these three lines do occur: other remarkable sets occur like $y + \bar{z} = 0$ (see [2, 3]). We call the fixed points, corresponding to this 'remaining' set of points, the 'R-type' points [25, 26].

One can use these localization properties to get, very quickly, a subset of all the fixed points, namely (for instance) the Q-type fixed points (one representative in the N-cycle belongs to line $y = (1 - \epsilon)/2$ or $z = (\epsilon - 1)/2$). These calculations can be performed quite efficiently since one can eliminate the y variable ($y = (1 - \epsilon)/2$) and, thus, reduce the calculations to looking for the roots (real or not) of an ϵ-dependent polynomial in this remaining z variable.
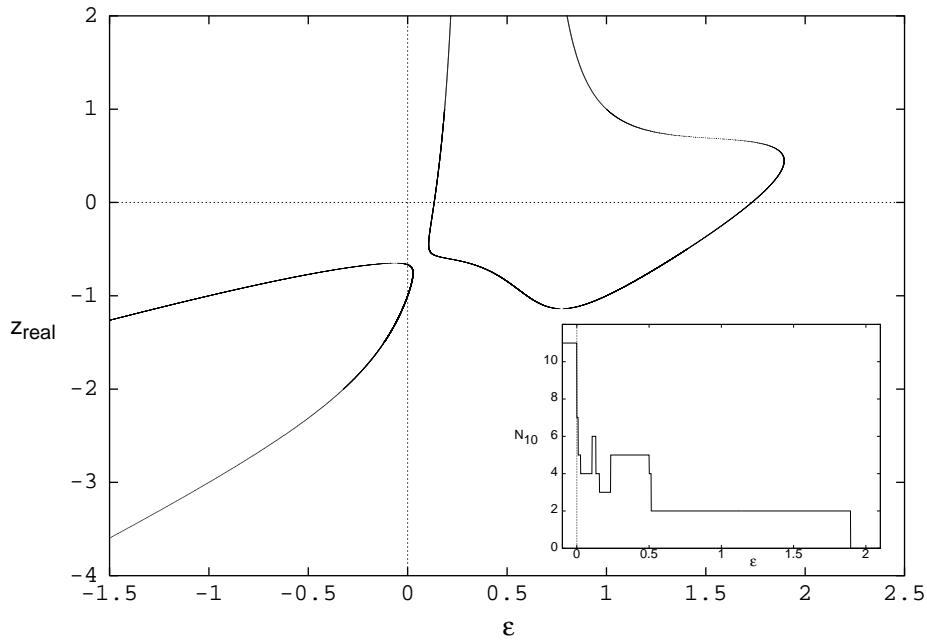
Figure 10. The real roots z of Q_{10}, as a function of ϵ, in the interval $[-1.5, 2.5]$. We have included the plot of the total number of fixed points of k_ϵ^{10} (P-type, Q-type and R-type) as a function of ϵ in an interval around $[0, 2]$.

One gets, for the first values of N, the following polynomial expressions relating z and ϵ:

$$Q_1(z, \epsilon) = 2 \cdot z - (\epsilon - 1) = 0 \qquad Q_3(z, \epsilon) = z - (\epsilon - 2) = 0$$
$$Q_5(z, \epsilon) = (3\epsilon - 1) \cdot z^2 - 2 \cdot (\epsilon - 3) \cdot (2\epsilon - 1) \cdot z + (\epsilon^3 - 5\epsilon^2 + 10\epsilon - 4) = 0, \ldots . \tag{37}$$

However, it can be seen that the cycles deduced here from Q_1, Q_3, Q_5, also have a representative on $y = -z$ and are thus also of the P-type. The first genuine Q-type cycles occur for $N = 8, 10, 12, \ldots$. In order to avoid any double-counting with P-type points, from now on we denote Q-type fixed points as the fixed points such that one representative in the N-cycle belongs to the line $y = (1 - \epsilon)/2$ or $z = (\epsilon - 1)/2$ but *none on $y = -z$*.

It is easy to see that the number of real roots z, of one of these $Q_N(z, \epsilon) = 0$ ($N = 8, 10, 12, \ldots$) conditions, varies with ϵ by intervals. The changes of this number of real roots take place at *algebraic* values of ϵ (resultant of $Q_N(z, \epsilon)$ in z). The details of the calculations, and a description of these polynomials, are given elsewhere [25, 26, 40]. The number of the fixed points of the Q-type (see [25, 40]) is, thus, a function of ϵ constant by interval, the limits of the intervals corresponding to some algebraic values (resultants deduced from the Q_N by eliminating z). For illustration, in figure 10 we plot the real roots: z, as a function of ϵ, for Q_{10}.

Let us give, for order 10, a few examples of these algebraic 'threshold' real values of ϵ corresponding to the real roots of such Q-type polynomials (37):

$$5\epsilon^2 - 10\epsilon + 1 = 0 \qquad 5\epsilon^4 - 96\epsilon^3 + 114\epsilon^2 - 40\epsilon + 1 = 0. \tag{38}$$

The roots of the first polynomial are of the form (40). The 'threshold' values of ϵ are thus given by two real roots $\epsilon \simeq 0.1055$ and 1.8944. The second polynomial only gives two real roots $\epsilon \simeq 0.027\,03$ and $\simeq 17.9549$.

Similar calculations can be performed for the fixed points of the P-type (see [25, 40]) corresponding to the line $y = -z$ (see [25, 26]). One can also get the real roots z of P_{10} as a function of ϵ. The algebraic values of ϵ, occurring in this case, are, again, the two roots of the first polynomial in (38) together with the only two real roots $\epsilon \simeq 0.1561$ and 0.5013 of the polynomial

$$\epsilon^8 - 26\epsilon^7 + 343\epsilon^6 - 2052\epsilon^5 + 6367\epsilon^4 - 7178\epsilon^3 + 3625\epsilon^2 - 824\epsilon + 64 = 0 \qquad (39)$$

as well as two real roots $\epsilon \simeq 0.008\,999$ and 0.1316 of a polynomial of degree 24 in ϵ that will not be presented here. The last set of points of the R-type (see [40]), which corresponds to fixed points that are neither of the P-type nor the Q-type, give the real roots $\epsilon \simeq 0.2338$, $0.514\,34$, and 33.2517, corresponding to polynomial $\epsilon^3 - 34\epsilon^2 + 25\epsilon - 4 = 0$. On all these algebraic values of ϵ, one can see a variation of the total number of fixed points (P-type, Q-type and R-type) of order 10 (see insertion in figure 10). These values are in fact particular examples of families of *algebraic* ϵ values. The simplest family of singled-out *algebraic* values of ϵ corresponds to the *fusion* of an N-cycle with the 1-cycle, and reads

$$\epsilon = \frac{1 - \cos(2\pi M/N)}{1 + \cos(2\pi M/N)} \qquad \text{or equivalently} \quad \cos(2\pi M/N) = \frac{1 - \epsilon}{1 + \epsilon} \quad (40)$$

for any integer N (with $1 < M < N/2$, M not a divisor of N). Other cycle-fusion mechanisms take place yielding new families of algebraic values for ϵ. For instance, the coalescence of the $(3 \times N)$-cycles in the 3-cycle and the coalescence of the $(4 \times N)$-cycles in the 4-cycle yield, respectively (with some constraints on the integer M that will not be detailed here),

$$\cos(2\pi M/N) = 1 - \frac{3}{4}\frac{\epsilon(\epsilon - 3)^2}{(1 - \epsilon)(1 + \epsilon)} \qquad \cos(2\pi M/N) = 1 - 32\frac{\epsilon(1 - \epsilon)^2}{(1 + \epsilon)^2(1 - 2\epsilon)}. \quad (41)$$

Status of the fixed points.   The fixed point of k_ϵ, which is elliptic for $\epsilon > 0$, becomes *hyperbolic* for $\epsilon < 0$. For three iterations ($N = 3$) one finds that moving through the $\epsilon = \frac{1}{3}$ value also changes the status of these fixed points from elliptic to hyperbolic. In fact, the algebraic values, like the ones depicted in (40) or (41), *also occur* in such changes of status from elliptic to hyperbolic (see [25, 26]). Therefore, the number of elliptic fixed points, or hyperbolic fixed points, is not as 'universal' as the total number of (complex) fixed points; however it has 'some universality': for a given value of N, the number of elliptic (resp. hyperbolic) fixed points depends on ϵ also by *intervals* (staircase function). This has, again, to be compared with the dependence of the growth complexity λ, seen as a function of ϵ, depicted in figure 1. The fact that the number of hyperbolic versus elliptic fixed points, as well as the number of *real versus nonreal* fixed points, is modified when ϵ goes through the *same set* of values, like (40) or (41), seems to indicate that a modification of the number of *real fixed points is not independent of the actual status of these points* (hyperbolic versus elliptic). This phenomenon, in fact, corresponds to a quite involved, and interesting, *structure* that will be sketched in [26].

4.3. Some expansions for the 'real dynamical zeta function' and H^{real}

Let us recall some results corresponding to $\epsilon = 0.52$, in particular the product decomposition (12) of the dynamical zeta function [2, 3]. In [2, 3], the number of irreducible cycles N_i (see (12)), as well as the number of irreducible cycles corresponding to hyperbolic points, elliptic points, real points is detailed (see table 1 in [3]). These results (and further calculations) enable us to write, for $\epsilon = 0.52$, the 'real dynamical function' $\zeta_\epsilon^{\text{real}}(t)$ as the following product:

$$\zeta_{52/100}^{\text{real}}(t) = \frac{1}{(1 - t)(1 - t^3)(1 - t^4)(1 - t^5)^2(1 - t^7)^2(1 - t^8)(1 - t^9)^4(1 - t^{10})^2}$$

$$\times \frac{1}{(1-t^{11})^6(1-t^{12})^{12}(1-t^{13})^{16}}\cdots \tag{42}$$

yielding the following expansion for $\zeta_\epsilon^{\text{real}}(t)$ and $H_\epsilon^{\text{real}}(t)$:

$$\begin{aligned}
\zeta_{52/100}^{\text{real}}(t) &= 1 + t + t^2 + 2t^3 + 3t^4 + 5t^5 + 6t^6 + 9t^7 + 13t^8 + 20t^9 + 28t^{10}\\
&\quad + 40t^{11} + 65t^{12} + 97t^{13} + \cdots\\
H_{52/100}^{\text{real}}(t) &= t + t^2 + 4t^3 + 5t^4 + 11t^5 + 4t^6 + 15t^7 + 13t^8 + 40t^9 + 31t^{10}\\
&\quad + 67t^{11} + 152t^{12} + 209t^{13} + \cdots.
\end{aligned} \tag{43}$$

The number of *real n*th cycles of the P-type, Q-type and R-type, denoted P_n, Q_n, and R_n respectively, are given in table B.4 in appendix B. For the R_n one cannot reduce, in contrast with the P-type or Q-type analysis, the calculations to a unique variable: one is obliged to perform a first resultant calculation where one eliminates one of the two variables and another resultant calculation where one eliminates the other one, and check back, in the Cartesian product of these possible solutions, the solutions which are actually fixed points. In order to get integer values that can be trusted, one needs to perform these (Maple) calculations with more than 2000 digits for order 12, but then one faces severe memory limitations in the formal calculations. We have been able to find integer values for the R_n for orders larger than 12; however it is clear that these integers are just lower bounds of the true integers (insufficient precision does not enable one to discriminate between fixed points that are very close). Therefore, we prefer not to give these integers here, and put a (*) in the tables in appendix B when we encounter these computer limitations.

The total number, T_n, of *real* cycles of the P-type, R-type and Q-type actually corresponds to the exponents in the product decomposition (42) for the 'real dynamical zeta function'. Unfortunately, these series are not large enough to 'guess' any possible (and simple, like (11)) rational expression for $\zeta^{\text{real}}(t)$, if any. Series (43), however, give a first 'rough estimate' for the 'real topological complexity' h_{real}: $h_{\text{real}} \simeq (97)^{1/13} \simeq 1.4217$ or else $h_{\text{real}} \simeq 209^{1/13} \simeq 1.508$, *clearly smaller* than the exact algebraic value for h corresponding to (11): $h \simeq 1.618\,03$.

Let us consider other values of ϵ.

For $\epsilon < 0$, one finds out that *all the fixed points seem to be real* and, thus, one can conjecture for $\epsilon < 0$ (but $\epsilon \neq -1$)

$$\zeta_{\epsilon<0}^{\text{real}}(t) = \frac{1-t^2}{1-t-t^2} \qquad \text{and} \qquad h_{\text{real}} = h \simeq 1.618\,03. \tag{44}$$

The number of cycles of the P-type, Q-type and R-type is given order by order in table B.1 in appendix B. These successive integer values for the total number of irreducible real cycles, T_n, yield

$$\begin{aligned}
\zeta_{\epsilon<0}^{\text{real}}(t) &= (1/((1-t)(1-t^3)(1-t^4)(1-t^5)^2(1-t^6)^2(1-t^7)^4\\
&\quad \times (1-t^8)^5(1-t^9)^8(1-t^{10})^{11}))\\
&\quad \times (1/((1-t^{11})^{18}(1-t^{12})^{25}(1-t^{13})^{40}(1-t^{14})^{58}(1-t^{15})^{90}\\
&\quad (1-t^{16})^{135}(1-t^{17})^{210}(1-t^{18})^{316}))\cdots.
\end{aligned} \tag{45}$$

With obvious notations, introducing 'restricted' dynamical zeta functions $\zeta_{\epsilon<0}^P(t)$ $\zeta_{\epsilon<0}^Q(t)$ and $\zeta_{\epsilon<0}^R(t)$ (corresponding to a product decomposition into irreducible cycles similar to (45), but where the cycles are of the P-type, Q-type and R-type, respectively), the real dynamical zeta function $\zeta_{\epsilon<0}^{\text{real}}(t)$ can be written as the product of these three expressions: $\zeta_{\epsilon<0}^{\text{real}}(t) = \zeta_{\epsilon<0}^P(t) \cdot \zeta_{\epsilon<0}^Q(t) \cdot \zeta_{\epsilon<0}^R(t)$. The restricted dynamical zeta function $\zeta_{\epsilon<0}^R(t)$ reads (see table B.1 in appendix B):

$$\begin{aligned}
\zeta_{\epsilon<0}^R(t) &= 1/((1-t^9)^2(1-t^{10})^2(1-t^{11})^6(1-t^{12})^{10}(1-t^{13})^{20}\\
&\quad \times (1-t^{14})^{30}(1-t^{15})^{60}(1-t^{16})^{88}(1-t^{17})^{156}(1-t^{18})^{238}\cdots).
\end{aligned} \tag{46}$$

Similar expressions can be written for $\zeta^P_{\epsilon<0}(t)$ and $\zeta^Q_{\epsilon<0}(t)$ (see table B.1 in appendix B). These restricted dynamical zeta functions do not seem to be the (product) expansions of any simple rational expression: *only their product is a remarkably simple rational expression*. When the number of iterations gets large, it seems that h_R, the growth rate of periodic points of the R-type, seems to dominate and thus $h \simeq h_R$. One has the following expansion for $\zeta^R_{\epsilon<0}(t)$ multiplied by the polynomial $(1 - t - t^2)$:

$$(1 - t - t^2) \cdot \zeta^R_{\epsilon<0}(t)$$
$$= 1 - t - t^2 + 2t^9 + 2t^{11} + 2t^{12} + 4t^{13} + 10t^{15} - 2t^{16} + 8t^{17} - 3t^{18} \cdots .$$

It might also be possible that the coefficients in the right-hand side of (47) could be bounded by a polynomial growth.

Remark. One can see the singled-out role played by these Q-type and P-type fixed points as related to the possible decomposition of (1) into the product of two involutions (reflections). From this decomposition, it can be seen that every intersection of the (globally) invariant set of points of one involution with the image a (globally) invariant set of points under some iterate is actually a periodic point. Therefore, one could suggest that there might be a simple relation between the Arnold complexity and the numbers of Q-type and P-type, rather than the total number of periodic points. One sees, for the $\epsilon < 0$ case, that this is probably not correct.

For $\epsilon = 3$, one has, at every order of iteration up to order 12, a *unique real* fixed point (the fixed point of order one but, of course, many complex fixed points) yielding

$$H_3^{\text{real}}(t) = \frac{t}{1 - t} \qquad \text{and} \qquad \zeta_3^{\text{real}}(t) = \frac{1}{1 - t} \tag{47}$$

and, for ϵ very close to 3, the expressions of $H^{\text{real}}(t)$ and $\zeta^{\text{real}}(t)$ cannot be distinguished (at the orders where we have been able to perform these fixed points calculations) from

$$H_{\epsilon\simeq3}^{\text{real}}(t) \simeq \frac{t \cdot (1 + t + 4t^2)}{1 - t^3} \qquad \text{and} \qquad \zeta_{\epsilon\simeq3}^{\text{real}}(t) \simeq \frac{1}{(1 - t) \cdot (1 - t^3)} \tag{48}$$

which just correspond to add an additional 3-cycle.

Expressions (47) are in agreement with the phase portrait of figure 9 for $\epsilon = 3$. This indicates that, seen as a mapping of two *real variables*, the mapping 'looks like' an integrable mapping: the 'real topological complexity' h_{real} seems to be exactly equal to 1 for $\epsilon = 3$ (and $h_{\text{real}} \simeq 1$ for $\epsilon \simeq 3$). The 'real topological entropy' $\log(h_{\text{real}})$ seems to be exactly zero for $\epsilon = 3$ and is, thus, drastically different from the generic 'usual' topological entropy $\log(1.618\,03\ldots)$. The possible foliation of the two-dimensional space in (transcendental) curves is discussed† elsewhere [41].

Miscellaneous examples are given in appendix B. In particular, the number of real cycles of the P-type, Q-type and R-type is given in table B.6 for $\epsilon = \frac{11}{10}$, yielding the following expansion for the real dynamical zeta function:

$$\zeta_{11/10}^{\text{real}}(t) = 1 + t + t^2 + 2t^3 + 2t^4 + 2t^5 + 3t^6 + 5t^7 + 5t^8 + 6t^9 + 10t^{10} + 12t^{11} + 13t^{12} + \cdots \tag{49}$$

clearly yielding a value for h_{real} close to 1 (e.g., $h_{\text{real}} \simeq (13)^{1/12} \simeq 1.238$) significantly smaller than $h \simeq 1.618$. This result has to be compared with the equivalent one for $\epsilon = \frac{9}{10}$ or for

† In particular it is shown that at least three of the (real) curves of the phase portrait correspond to *divergent series* satisfying *an exact functional equation* [41].

$\epsilon = \frac{21}{25}$:

$$\zeta_{21/25}^{\text{real}}(t) = \frac{1}{(1-t)(1-t^3)(1-t^4)(1-t^7)^2(1-t^8)(1-t^{10})^2(1-t^{11})^4(1-t^{12})^2} \cdots$$
$$= 1 + t + t^2 + 2t^3 + 3t^4 + 3t^5 + 4t^6 + 7t^7$$
$$+ 9t^8 + 10t^9 + 15t^{10} + 23t^{11} + 28t^{12} + \cdots \tag{50}$$

yielding a larger value for h_{real}: $h_{\text{real}} \simeq (28)^{1/12} \simeq 1.32$. This expansion is actually compatible with the following simple rational expression and for its logarithmic derivative $H_{21/25}^{\text{real}}$:

$$\zeta_{21/25}^{\text{real}}(t) = \frac{1+t^2}{1-t+t^2-2t^3} \qquad \text{and} \qquad H_{21/25}^{\text{real}} = \frac{t(5t^2+2t^4+1)}{(1+t^2)\cdot(1-t+t^2-2t^3)}. \tag{51}$$

Note that all the coefficients of the expansion of the rational expression (51) and of its logarithmic derivative $H_{21/25}^{\text{real}}$ are *positive* (in contrast with a Padé approximant (75) given in appendix C for $\epsilon = \frac{2}{3}$ which is ruled out because coefficient t^{54} of its expansion is negative). If this simple rational expression is actually the exact expression for the real dynamical zeta function $\zeta_{21/25}^{\text{real}}(t)$ this would yield the following algebraic value for h_{real}: $h_{\text{real}}(\frac{21}{25}) \simeq 1.353\,21$.

For $\epsilon = \frac{9}{10}$, one gets the same product decomposition, at least up to order ten. The number of nth cycles of the P-type, Q-type and R-type for $\epsilon = \frac{9}{10}$ are given in appendix B. One thus sees that h_{real} decreases when ϵ crosses the $\epsilon = 1$ value.

For $\epsilon = \frac{1}{4}$ we have obtained (see appendix B)

$$\zeta_{1/4}^{\text{real}}(t) = 1/((1-t)(1-t^3)(1-t^4)(1-t^5)^2(1-t^7)$$
$$\times (1-t^8)(1-t^9)^3(1-t^{10})^2(1-t^{11})^4(1-t^{12})^4(1-t^{13})^8 \cdots)$$
$$= 1 + t + t^2 + 2t^3 + 3t^4 + 5t^5 + 6t^6 + 8t^7 + 12t^8$$
$$+ 18t^9 + 25t^{10} + 34t^{11} + 48t^{12} + 70t^{13} + \cdots. \tag{52}$$

The 'nongeneric' values $\epsilon = 1/m$ and $(m-1)/(m+3)$ require a special and careful analysis†. However, as was seen for $\zeta(t)$, one clearly verifies, on all these 'real dynamical zeta function' $\zeta^{\text{real}}(t)$, that the coefficients in these expansions are continuous in ϵ near these points except at these very values of ϵ where one gets smaller integers and, possibly, smaller values for h_{real} (the limit at the left and on the right of h_{real} are equal and larger than h_{real} on these very 'nongeneric' values).

The numbers of irreducible *real* n-cycles of the P-type, Q-type and R-type are given in appendix B for miscellaneous values of ϵ: $\epsilon = \frac{11}{100}, 5, 10, 50$. We also give, in appendix C, for various values of ϵ ($\epsilon = \frac{9}{50}, \frac{31}{125}, \frac{12}{25}, \frac{17}{25}, \frac{66}{125}, \frac{3}{4}, \frac{3}{2}$), the product decomposition and expansions for $\zeta_\epsilon^{\text{real}}(t)$ up to order 11.

Similar calculations of the expansions of $H_\epsilon^{\text{real}}(t)$ and $\zeta_\epsilon^{\text{real}}(t)$, for many other values of the parameter ϵ, have been performed. For most of these other values of ϵ the series are not large enough to 'guess' a rational expression (if any) for the 'real dynamical zeta function' $\zeta^{\text{real}}(t)$; however all these results confirm that h_{real} varies with ϵ when ϵ is positive, while h_{real} is constant (except $\epsilon = -1$) when $\epsilon < 0$. When ϵ is positive, the estimates of h_{real} are in agreement with the 'visual complexity' as seen on the phase portraits (see the previous section). In particular, one finds that h_{real} roughly decreases as a function of ϵ in the intervals $[0^+, \simeq \frac{1}{10}]$ and $[\simeq \frac{1}{3}, 1^-]$, and increases in the interval $[\simeq \frac{1}{10}, \simeq \frac{1}{3}]$ (with a sharp decrease near $\epsilon \simeq \frac{1}{2}$ and $\epsilon \simeq 1$), that h_{real} is close or very close to one when ϵ belongs to an interval $[1^+,$

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† Some fixed points near these 'nongeneric' values ($\epsilon \simeq 1/m$) disappear on these very values *stricto sensu*: they become singular. One has to verify carefully that all the points obtained in such calculations are fixed points and not singular points.

$\simeq 16]$ and that h_{real} grows monotonically with ϵ for $\epsilon > 16$ to reach some asymptotic value in the $\epsilon \to \infty$ limit. It will be seen, in the next section, that the 'real topological complexity' h_{real}, in the $\epsilon \to \infty$ limit, tends to a value $h_{\text{real}} \simeq 1.429$ clearly different, again, from the generic 'topological complexity' $h \simeq 1.618$.

4.4. Seeking for rationality for the 'real dynamical zeta function'

Recalling the large number of rational expressions obtained for the dynamical zeta functions [2, 3] and the degree generating functions [2, 4, 6], one may have a rationality 'prejudice' for these 'real dynamical zeta functions' $\zeta_\epsilon^{\text{real}}(t)$, calculated for a *given* value of ϵ. However, the occurrence of any symbolic dynamic, and associated Markov partition, is *far from being natural in this real analysis framework* [27–29]. If one bets on the rationality of the real dynamical zeta function $\zeta_\epsilon^{\text{real}}(t)$ (see (33)), it must, however, be clear that $\zeta_\epsilon^{\text{real}}(t)$ depends on ϵ in a very complicated way (piecewise continuous functions, devil's staircase?). If, for some given value of the parameter ϵ, the partial dynamical zeta function $\zeta_\epsilon^{\text{real}}(t)$ actually corresponds to a rational expression, one should, in fact, have an *infinite set* of such rational expressions associated with the infinite number of steps (intervals† in ϵ) in the 'devil's staircase'. The actual location of these 'steps', that is, the limits of these intervals in ϵ, corresponds to an infinite number of values of ϵ like (40) or (41) (and others [26]). For a given ϵ, the calculations of the first terms of the expansion of the 'real dynamical zeta function' $\zeta_\epsilon^{\text{real}}(t)$ do not rule out, at the order for which we have been able to perform these calculations (10, 11), *rational expressions* (see, for instance, [41]).

The number of *real* nth cycles of the P-type, Q-type and R-type, for $\epsilon = 50$, is given in table B.9 in appendix B. At order 11, the number of irreducible *real* cycles, and therefore the expansion of the 'real dynamical zeta function' are the same for $\epsilon = 50, 100, 1000, \ldots$. For $\epsilon = 50, 100$ one has a product expansion for the dynamical zeta function identical, up to order 11, to the product expansion corresponding to the large-ϵ limit (see (53) below). These expansions are, however, different at order 12, see appendix B.

For ϵ large enough, one gets the following cycle product decomposition:

$$\zeta_{\epsilon=\infty}^{\text{real}}(t) = 1/((1-t)(1-t^3)(1-t^5)^2(1-t^7)^2(1-t^8)^2$$
$$\times (1-t^9)^2(1-t^{10})^3(1-t^{11})^4(1-t^{12})^6 \cdots) \tag{53}$$

corresponding to the number of *real* P-type, Q-type and R-type n cycles for $\epsilon = 20\,000$ given in table B.11 in appendix B. Note that one gets the same table (up to order 16) for $\epsilon = 1000, 100\,000, 1000\,000$.

One finds out easily that these results, for the 'real dynamical zeta function' $\zeta^{\text{real}}(t)$, are (up to order 12) actually in *perfect agreement* with (the expansion of) the rational expression

$$\zeta_{\epsilon=\infty}^{\text{real}}(t) = \frac{1+t}{1-t^2-t^3-t^5} = \frac{1-t^2}{(1-t-t^2)+t^4 \cdot (1-t+t^2)} \tag{54}$$

yielding an algebraic value for h_{real}: $h_{\text{real}} \simeq 1.4291$. If one 'believes' in some symbolic dynamic coding interpretation, or in the existence of a linear transfer operator‡, matrix A, such that the denominator of (54), $1-t^2-t^3-t^5$, can be written as $\det(Id - t \cdot A)$, one finds

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† In contrast with the situation for the 'standard' dynamical zeta function which is equal to one generic universal expression (like (11)), up to a (zero measure) set of values of ϵ (see figure 1).
‡ For more details on linear transfer operators in a symbolic dynamics framework see, e.g., [19, 21–23].

that a possible choice for this transition matrix is

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}. \tag{55}$$

In contrast with a Markov transition matrix, this matrix is not such that the sums of the entries in each row, or column, are equal.

5. Real Arnold complexity

As with the topological entropy, the Arnold complexity can be 'adapted' to define a '*real Arnold complexity*'. The Arnold complexity counts the number of intersections between a fixed (complex projective) line and its Nth iterate [5]: let us now count the number of *real* points which are the intersections between a *real fixed line* and its Nth iterate. With this restriction to *real* points we have lost 'most of the universality properties' of the 'usual' (complex) Arnold complexity. For various values of ϵ, we have calculated the number of intersections of various (real) lines with their Nth iterates. In contrast with the 'usual' Arnold complexity [5], which does not depend on the (complex) line one iterates (topological invariance [24]), it is clear that the number of real intersections depends on the chosen line, but one can expect that the *asymptotic behaviour* of these numbers for N large enough will not depend too much of the actual choice of the (real) line one iterates. Actually, we have discovered for this very example that this seems to be the case (except for some nongeneric lines). Furthermore, the real line $y = (1 - \epsilon)/2$, which is known to play a particular role for mapping (1) (see section 4.2), is very well suited to perform this number of intersection calculations†: for this particular line the successive numbers of intersections are extremely regular, thus enabling one to better estimate this asymptotic behaviour λ_{real}^N of the 'real Arnold complexity'; but, of course, similar calculations can be performed with an arbitrary (generic) line. The quantity λ_{real} can be seen as the equivalent, for *real mappings*, of the growth complexity λ (see section 2.2 and [4]). Let us try to obtain λ_{real} for various values of ϵ.

As for the semi-numerical method detailed in section 2.2, we have again developed a C-program using the multiprecision library gmp [14], counting the number of (real) intersections of the $y = (1 - \epsilon)/2$ real line (for instance) with its Nth iterate. This program does not calculate the precise location of the intersection points: it is based on *Sturm's theorem*‡. All these calculations have been cross-checked by a (Maple) program calculating the numbers of intersections using the sturm procedure in Maple§. The results of these calculations are given in figure 11.

Let us denote by $\mathcal{A}_N$ the number of (real) intersections for the Nth iterate. In order to estimate 'real growth complexity' λ_{real} we have plotted $\mathcal{A}_N^{1/N}$, for various values of the number

† Again, these calculations could (and actually have) been performed with another real line. We have just found experimentally that this very line yields more regular results and, furthermore, enables one to perform the calculations at a higher order.

‡ Assuming that a polynomial $P(x)$ has no multiple roots, one can build a finite series of polynomials corresponding to the successive Euclidean division of $P(x)$ by its first derivative $P'(x)$. See, for instance, [42] for more details on the Sturm sequences and Sturm's theorem.

§ The sturm procedure one can find in Maple gives the number of real roots of a polynomial in any interval $[a, b]$, even the interval $[-\infty, +\infty]$. The procedure sturm uses *Sturm's theorem* to return the number of real roots of polynomial P in the interval $[a, b]$. The first argument of this sturm procedure is a *Sturm sequence* for P, which can be obtained with another procedure, sturmseq, which returns the Sturm sequence as a list of polynomials and replaces multiple roots by single roots.
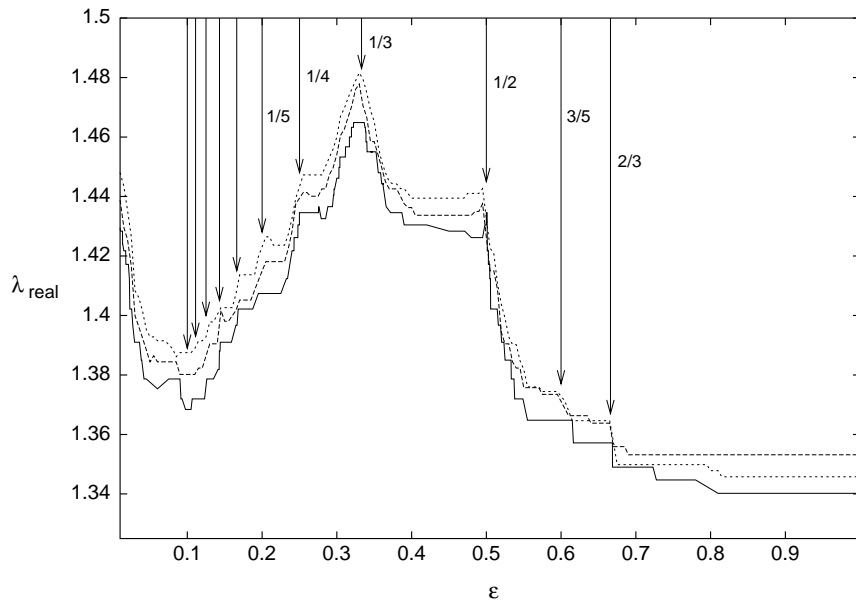
Figure 11. A representation of $\lambda_{\text{real}}(\epsilon)$ by $\mathcal{A}_N^{1/N}$, as a function of ϵ, in the $[0, 1]$ interval, for $N = 13$ (full curve) 14 (dashed curve), and 15 (dotted curve). The singled-out $\epsilon = 1/m$ values for $m = 2, 3, 4, 5, \ldots$ and $\epsilon = (m-1)/(m+3)$ for $m = 7, 9, \ldots$ clearly play a special role for these various 'staircase functions' in the large-N limit.

of iterations ($N = 13, 14, 15$), as a function of ϵ, in the range $[0, 1]$ where λ_{real} has a quite 'rich' behaviour.

This behaviour should be compared with the 'universal' behaviour shown in figure 1. From figure 11, one sees that the singled-out values $\epsilon = 1/m$, as well as $\epsilon = (m-1)/(m+3)$ for $m = 7, 9, \ldots$, seem, again, to play a special role in the large-$N$ limit. Of course, recalling the results of section 4.2, it is clear that $\mathcal{A}_N^{1/N}$ is a staircase function of ϵ, for N finite, the limits of each interval corresponding to *algebraic values* (like (40)) sketched in sections 4.3, 4.2. These algebraic values form an *infinite set* of values which accumulate everywhere in the $[0, 1]$ interval. What is the limit of these functions $\mathcal{A}_N^{1/N}(\epsilon)$ when N becomes large: a devil's staircase or a (piecewise) continuous function? The 'shape' of $\mathcal{A}_N^{1/N}$, as a function of ϵ, is 'monotonic enough' (see figure 12) in different intervals, namely $\epsilon < 0$, and in the intervals of ϵ roughly given by $[0^+, \simeq 1/10]$, $[\simeq \frac{1}{10}, \simeq \frac{1}{3}]$, $[\simeq \frac{1}{3}, 1^-]$, $[1^+, \simeq 16.8]$ and $[\simeq 16.8, \infty]$, such that one may expect that the infinite accumulation of these algebraic values (like (40)) could yield a perfectly continuous function $\lambda_{\text{real}}(\epsilon)$ (except on the nongeneric values $\epsilon = 1/m$ and $\epsilon = (m-1)/(m+3)$) and not a devil's staircase-like function. This question remains open at the present moment. When ϵ varies from $-\infty$ to ∞ the behaviour of the 'real growth complexity' λ_{real}, as a function of ϵ, is not as 'rich' as in the interval $[0^+, 1^-]$ depicted in figure 11. One finds that λ_{real} is close, or extremely close to, 1 in quite a large interval $[1^+, \simeq 16.8]$ and that it increases monotonically with ϵ in the $[\simeq 16.8, \infty]$ interval to reach some asymptotic value in the $\epsilon \to \infty$ limit. In fact, a logarithmic scale in ϵ is better suited to describe λ_{real} as a function of ϵ. Figure 12 represents λ_{real}, more precisely $\mathcal{A}_{13}^{1/13}$, as a function of $\log(2+\epsilon)$. For $\epsilon = -1, 0, \frac{1}{3}, \frac{1}{2}, 1$ we know that λ_{real} will be exactly equal to 1 (integrable cases [8]). On these points (represented by squares in figure 12), as well as on the $\epsilon = 1/m$ and $\epsilon = (m-1)/(m+3)$ nongeneric points, λ_{real} is *not continuous* as a function of ϵ. We have not represented these
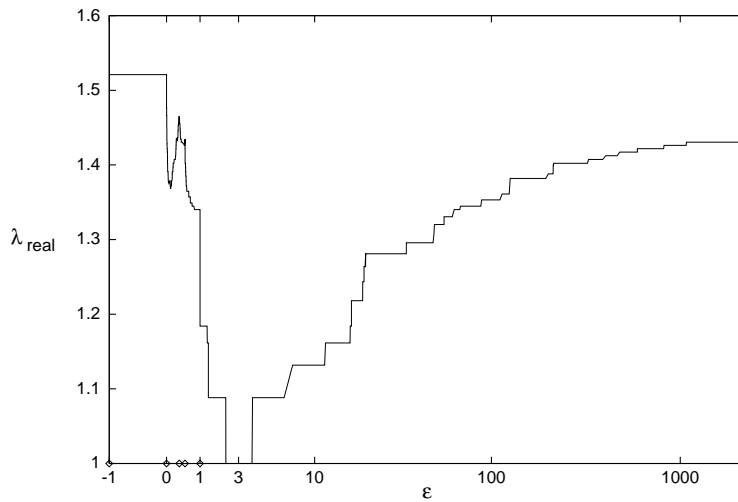
Figure 12. A representation of λ_{real} by $\mathcal{A}_{13}^{1/13}$, as a function of ϵ, in a $\log(2 + \epsilon)$ logarithmic scale. The integrable points $\epsilon = -1, 0, \frac{1}{3}, \frac{1}{2}, 1$ are represented by squares.

other nongeneric points. They have to be calculated separately.

A first estimate of λ_{real}, for ϵ large, is $\lambda_{\text{real}} \simeq (214)^{1/15} \simeq 1.430\,08$. We are now ready to compare the 'real topological entropy' and the 'real Arnold complexity' for different values of ϵ, and see if the identification between h, (the exponential of) the topological entropy, and λ, characterizing the (multiplicative rate of growth of the) usual Arnold complexity, also holds for their 'real partners', namely h_{real} and λ_{real}. Actually, one finds that this identification (which is obviously true for $\epsilon < 0$) also holds for $\epsilon = 3$. One gets numerical results for various values of ϵ (for which we have estimated the 'real' topological entropy $\log(h_{\text{real}})$ (see section 4.3)) *quite compatible with this identification*. In particular, for ϵ large, we see that these two 'real complexities' *give extremely close results*, namely $h_{\text{real}} \simeq 1.4291$ and $\lambda_{\text{real}} \simeq 1.43$.

6. Real Arnold complexity generating functions: seeking rationality

As with the introduction of the 'real dynamical zeta functions' $\zeta^{\text{real}}(t)$, one can introduce the generating function of the previous 'real Arnold complexities' $\mathcal{A}_N$:

$$\mathcal{A}_\epsilon(t) = \sum_N \mathcal{A}_N \cdot t^N. \tag{56}$$

Recalling the large number of rational expressions obtained for the dynamical zeta functions [2, 3] and the degree generating functions [2, 4, 6], one may have, again, a rationality 'prejudice' for these 'real Arnold complexity generating functions'. Let us try to see if the expansions of these generating functions $\mathcal{A}_\epsilon(t)$ coincide, for some given values of ϵ, with the expansion of some (hopefully simple) rational expressions. For *any negative value* of ϵ (except $\epsilon = -1$, see (63)) the expansion of the 'real Arnold complexity' generating function $\mathcal{A}_\epsilon(t)$ is equal, up to order 15, to

$$\mathcal{A}_{\epsilon < 0}(t) = \frac{t}{1 - t - t^2} \tag{57}$$

in agreement with (44). The $h = \lambda$ equality being probably not valid in general (see (30), (32)), one should not expect, in the most general birational framework, a similar equality to hold for

h_{real} and λ_{real}. We have just compared h_{real} and λ_{real} for a family of mappings for which the $h = \lambda$ equality seems to be valid. If an equality between two 'real' complexities exists, they are clearly the best candidates: they are both invariant under conjugations by any arbitrary real function. The fact that for any $\epsilon < 0$ ($\epsilon \neq -1$) $h_{real} = h$ and $\lambda_{real} = \lambda$ strongly suggests that the equality $h_{real} = \lambda_{real}$ could be valid for any ϵ.

In order to compare more carefully h_{real} and λ_{real}, and find some possible nontrivial rational expressions for $\mathcal{A}_\epsilon(t)$, let us give, in the following, miscellaneous expansions of $\mathcal{A}_\epsilon(t)$ for various values of ϵ.

6.1. Expansions for the 'real Arnold complexity' generating functions

In fact, we have not only calculated the real Arnold complexities $\mathcal{A}_{13}$, $\mathcal{A}_{14}$ and $\mathcal{A}_{15}$ required to plot figures 11 and 12, but actually obtained all the coefficients, up to order 13, for 2000 values of ϵ, and up to order 15 for 200 values of ϵ. We thus have the expansion of $\mathcal{A}_\epsilon(t)$ up to order 13 (resp. 15) for several thousand values of ϵ.

The expansion of $\mathcal{A}_\epsilon(t)$ for $\epsilon = 0.52$ for which the expansion of the real dynamical zeta function has been given previously (see (42) and (43)). One gets the following expansion:

$$\mathcal{A}_{52/100}(t) = t + t^2 + 2t^3 + 3t^4 + 5t^5$$
$$+ 6t^6 + 11t^7 + 11t^8 + 16t^9 + 29t^{10} + 33t^{11} + 46t^{12} + 73t^{13} + \cdots. \tag{58}$$

This series yields a first rough approximation of λ_{real} corresponding to $\lambda_{real} \simeq (73)^{1/13} \simeq 1.391$, clearly smaller than the generic complexity $\lambda \simeq 1.618\,03$, and in good enough agreement with the estimation of h_{real} one can deduce from (43), namely $h_{real} \simeq (93)^{1/13} \simeq 1.417$. Of course, these two series are too short to see if an identity like $h_{real} = \lambda_{real}$ really holds.

Considering h_{real} as a function of ϵ, it is clear that the general shape of this 'curve' looks extremely similar to the curve corresponding to λ_{real} as a function of ϵ (see figures 11 and 12); it is also constant for ϵ negative, gets very close to 1 around $\epsilon \simeq 3$, grows monotonically for $\epsilon > 10$ and tends to a nontrivial asymptotic value $h_{real} \simeq 1.4291$.

Therefore, in order to get some hint of a possible $h_{real} = \lambda_{real}$ identity, it is necessary to see if this relation holds for various values of ϵ for which h_{real} and λ_{real} can be calculated exactly or for which very good approximations can be obtained: namely $\epsilon < 0$, all the integrable values, or $\epsilon = 3$ and its neighbourhood.

For $\epsilon = 3$, the 'real Arnold complexity' generating function $\mathcal{A}_\epsilon(t)$ is equal, up to order 15, to the simple *rational* expression

$$\mathcal{A}_3(t) = \frac{t}{1-t} \tag{59}$$

which is in perfect agreement with the result for the 'real dynamical zeta function' $\zeta_{\epsilon=3}^{real}(t)$ (see (47)). For ϵ very close to 3 one gets

$$\mathcal{A}_{\epsilon \simeq 3}(t) \simeq \frac{t}{1-t} + \frac{t^3}{1-t^3} = \frac{t \cdot (1 + t + 2t^2)}{1 - t^3} \tag{60}$$

again in good agreement with (48).

Integrable values for ϵ. For $\epsilon = \frac{1}{2}$, the generating function for the 'real Arnold complexity', $\mathcal{A}_\epsilon(t)$, is equal, up to order 38, to the expansion of the rational expression

$$\mathcal{A}_{1/2}(t) = \frac{t(1-t^7)}{(1-t)^2(1-t^5)(1+t)} + \frac{t^4(1-t^9)}{(1-t)(1-t^5)(1-t^6)(1+t)}$$
$$+ \frac{t^5}{(1-t^5)(1-t^3)(1+t)} + 2\frac{t^{28}}{1-t^5} \tag{61}$$

to be compared with $A_{1/2}(t)$ given in (5).

For $\epsilon = \frac{1}{3}$ the calculations corresponding to the generating function for the 'real Arnold complexity' are, in contrast, quite trivial, yielding

$$\mathcal{A}_{1/3}(t) = \frac{t \cdot (1+t)}{1-t^3} = A_{1/3}(t). \tag{62}$$

For $\epsilon = -1$ the generating function for the 'real Arnold complexity' is equal, up to order 15, to the rational expression

$$\mathcal{A}_{-1}(t) = \frac{t}{1-t^2} = A_{-1}(t). \tag{63}$$

For $\epsilon = +1$ the generating function for the 'real Arnold complexity' is equal, up to order 15, to the rational expression

$$\mathcal{A}_1(t) = \frac{t}{(1-t^2) \cdot (1-t)} = A_1(t). \tag{64}$$

All these results have to be compared with the generating functions (5).

Nongeneric values for ϵ. The nongeneric values of ϵ require special attention. For instance, for $\epsilon = \frac{1}{4}$ one obtains the following expansion†:

$$\mathcal{A}_{1/4}(t) = t + t^2 + 2t^3 + 3t^4 + 5t^5 + 6t^6 + 8t^7 + 11t^8 + 17t^9 + 23t^{10}$$
$$+ 31t^{11} + 44t^{12} + 63t^{13} + 90t^{14} + 128t^{15} + 183t^{16} + \cdots$$

and for $\epsilon = \frac{1}{5}$ one gets

$$\mathcal{A}_{1/5}(t) = t + t^2 + 2t^3 + 2t^4 + 4t^5 + 4t^6 + 6t^7 + 7t^8 + 12t^9 + 15t^{10}$$
$$+ 19t^{11} + 28t^{12} + 33t^{13} + 53t^{14} + 77t^{15} + \cdots. \tag{65}$$

Since $\epsilon = \frac{1}{5}$ is a nongeneric value (it is of the form $1/m$), the previous expansion (65) can be compared with the ones corresponding to values very close to $\frac{1}{5}$ but not equal: for instance $\epsilon = \frac{99}{500}$ and $\epsilon = \frac{101}{500}$ which are given in appendix D, or, closer to $\epsilon = \frac{1}{5}$ $\epsilon = \frac{999}{5000}$ and $\epsilon = \frac{1001}{5000}$:

$$\mathcal{A}_{999/5000}(t) = \mathcal{A}_{1001/5000}(t) = t + t^2 + 2t^3 + 3t^4 + 5t^5 + 6t^6$$
$$+ 9t^7 + 13t^8 + 18t^9 + 27t^{10} + 41t^{11} + 66t^{12} + 89t^{13} + \cdots. \tag{66}$$

Similar expansions, corresponding to values close to the nongeneric value $\epsilon = \frac{1}{4}$, are given in appendix D. All these results show that, as with the situation for the customary topological entropy or the growth complexity λ (see figure 1), λ_{real} is continuous as a function of ϵ *near* the nongeneric values of $\epsilon \simeq 1/m$; however, exactly on these very nongeneric values λ_{real} takes *smaller* values (continuous function up to a zero measure set).

Remark. It is natural to compare the expansion corresponding to $\epsilon = \frac{3}{5}$ with the one corresponding to $\epsilon = \frac{1}{7}$, since $\epsilon = \frac{1}{7}$ and $\epsilon = \frac{3}{5}$ have the *same topological entropy* (growth complexity λ) associated with $1 - t - t^2 + t^{m+2}$ for $m = 7$ (see relation (13)). One gets for $\epsilon = \frac{3}{5}$

$$\mathcal{A}_{3/5}(t) = t + t^2 + 2t^3 + 3t^4 + 2t^5 + 5t^6 + 9t^7$$
$$+ 8t^8 + 11t^9 + 14t^{10} + 18t^{11} + 24t^{12} + 29t^{13} + 41t^{14} + 51t^{15} + \cdots$$

and for $\epsilon = \frac{1}{7}$

$$\mathcal{A}_{1/7}(t) = t + t^2 + 2t^3 + t^4 + 4t^5 + 7t^6 + 7t^7$$
$$+ 8t^8 + 13t^9 + 16t^{10} + 22t^{11} + 36t^{12} + 43t^{13} + 65t^{14} + 87t^{15} + \cdots.$$

† These Maple calculations have been performed with 6000 digits, but they are already stable with 2000 digits.

These two expansions *do not seem to yield the same value for* λ_{real} ($\lambda_{\text{real}}(\frac{3}{5}) \simeq 51^{1/15} \simeq 1.2997$ and $\lambda_{\text{real}}(\frac{1}{7}) \simeq 87^{1/15} \simeq 1.3468$) *although they share the same growth complexity* λ.

The expansions of $\mathcal{A}_\epsilon(t)$ near $\epsilon = \frac{3}{5}$ are given in appendix D.

Miscellaneous values of ϵ.   For most of the values of ϵ the expansions are not long enough to 'guess' rational expressions (if any). One can, however, get some estimates of λ_{real} that can be compared with h_{real}.

For $\epsilon = \frac{21}{25}$ one gets the following result:

$$\mathcal{A}_{21/25}(t) = t + t^2 + 2t^3 + 3t^4 + 3t^5 + 6t^6$$
$$+ 7t^7 + 11t^8 + 12t^9 + 21t^{10} + 25t^{11} + 36t^{12} + 45t^{13} + 69t^{14} + \cdots .$$

This expansion seems to yield the following estimated value for λ_{real}: $\lambda_{\text{real}}(\frac{21}{25}) \simeq 69^{1/14} \simeq 1.3531$, to be compared with (50). In fact, this expansion is *actually compatible* with the expansion of the rational expression

$$\mathcal{A}_{21/25}(t) = \frac{t \cdot (1 + t + t^2 + t^3 - 2t^4)}{(1 - t)(1 + t)^2(1 - t + t^2 - 2t^3)}. \tag{67}$$

Note that the rational expression (67) has actually the *same singularity* that the rational expression (51) suggested for the real dynamical zeta function $\zeta^{\text{real}}_{21/25}(t)$. All the coefficients of the expansion of (67) are *positive* (in contrast to (75) given in appendix C which is ruled out because coefficient t^{54} of its expansion is negative). If this simple rational expression were actually the exact expression for $\mathcal{A}_{\frac{21}{25}}(t)$ it would yield the following algebraic value for λ_{real}: $\lambda_{\text{real}}(\frac{21}{25}) = h_{\text{real}}(\frac{21}{25}) \simeq 1.35321$.

Some results for ϵ larger than 1 (again obtained with 6000 digits) are given in appendix D ($\epsilon = \frac{3}{2}, \epsilon \simeq 2$ and $\epsilon = 4, 5, 6, 10, 20, 30$). These series indicate that an estimated value for λ_{real} could correspond to λ_{real} very close to 1 for $\epsilon = \frac{3}{2}, \epsilon \simeq 2$, and $\epsilon = 4, \ldots, 10$, and even quite close to 1 for $\epsilon = 20$.

6.2. 'Real Arnold complexity' generating functions for ϵ large

Examination of figure 12 shows that λ_{real} goes to some nontrivial limit, $\lambda_{\text{real}} \simeq 1.429$, in the large-$\epsilon$ limit. Let us examine the expansion of $\mathcal{A}_\epsilon(t)$ for various increasing values of ϵ, in order to study this $\epsilon \to \infty$ limit. The expansions of $\mathcal{A}_\epsilon(t)$ for $\epsilon = 40, 50, 100, 500, 1000$ are given in appendix D, up to order 13.

For ϵ large the expansion of $\mathcal{A}_\epsilon(t)$, the generating function for the 'real Arnold complexity', is equal, up to order 15, to (for instance† for $\epsilon = 20\,000$)

$$\mathcal{A}_{20\,000}(t) = t + t^2 + 2t^3 + 3t^4 + 5t^5 + 8t^6 + 11t^7 + 17t^8 + 24t^9$$
$$+ 35t^{10} + 51t^{11} + 72t^{12} + 105t^{13} + 149t^{14} + 214t^{15} + \cdots \tag{68}$$

which actually coincides with the expansion of the simple rational expression

$$\mathcal{A}_\infty(t) = \frac{t \cdot (1 + t^4)}{(1 - t^2 - t^3 - t^5) \cdot (1 - t)} = \frac{t \cdot (1 + t^4)}{(1 - t - t^2) + t^4 \cdot (1 - t + t^2)}. \tag{69}$$

This last result has to be compared with the equivalent one for the 'real dynamical zeta function' $\zeta^{\text{real}}_{\epsilon=\infty}(t)$ (see (54) in section 4.4). These two nontrivial rational results, for ϵ large, are in *perfect*

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† These calculations have to be performed with at least 6000 digits. With a number of lower than 2000 digits one gets *smaller* coefficients: the precision is not large enough to distinguish between very close intersection points.

agreement, yielding the same *algebraic value* for the two 'real complexities' h_{real} and λ_{real}, namely $h_{\text{real}} = \lambda_{\text{real}} \simeq 1.429\,108\,32$.

All the results displayed in this section seem to show that the identification between h_{real} and λ_{real} actually holds.

Remark. Recalling the 'universal' relation (20), or more precisely (21), which gives (for ϵ generic and for $\epsilon = 1/m$, $m \geqslant 4$, $\epsilon = (m+1)/(m+3)$ for $m = 9, 13, 17, \ldots$) a ratio $\zeta_\epsilon(t)/A_\epsilon(t)$ equal to $(1 - t^2)/t$, one can look at the 'real ratio' $\zeta_\epsilon^{\text{real}}(t)/\mathcal{A}_\epsilon(t)$. Of course, for $\epsilon < 0$, this 'real ratio' is also equal to $(1 - t^2)/t$; however in the $\epsilon \to \infty$ limit it tends to be equal to $(1 - t^2)/t/(1 + t^4)$. Therefore one should not expect any simple 'universal' relation like (20) between $\zeta_\epsilon^{\text{real}}(t)$ and $\mathcal{A}_\epsilon(t)$.

These various Arnold complexity generating functions $\mathcal{A}_\epsilon(t)$ were associated with the iteration of the (real or complex) line $y = (1 - \epsilon)/2$. One can introduce a generating function for each line (or fixed curve) one iterates. The corresponding series become slightly more difficult to extrapolate but give similar results, particularly the asymptotic values for λ_{real}. The sensitivity of the previous analysis, according to the chosen curve one iterates, will be discussed elsewhere.

It would be interesting to see if the 'real dynamical zeta functions' $\zeta_\epsilon^{\text{real}}(t)$ or the 'real degree generating functions' $\mathcal{A}_\epsilon(t)$ might also be rational expressions for *other* values of ϵ or, even, if these 'real generating functions' might be rational expressions for *any given* value of ϵ. In this last case there should be an infinite number of such *rational* expressions: it is clear that they could not all be 'simple' like (54) or (69).

7. Conclusion

The results presented here seem to be in agreement with, again, an identification between λ_{real}, the (asymptotic) '*real Arnold complexity*', and h_{real}, the (exponential of the) '*real topological entropy*' for the very example of two-dimensional mapping analysed in this paper. In contrast with the 'universal' behaviour of the 'usual' Arnold complexity, or topological entropy, displayed in figure 1, λ_{real} and h_{real} are quite involved functions of the parameter ϵ, which the birational transformations depend on (see figures 11 and 12).

We have, however, obtained some remarkable rational expressions for the real dynamical zeta function $\zeta_\epsilon^{\text{real}}(t)$ and for a 'real Arnold complexity' generating function $\mathcal{A}_\epsilon(t)$. In particular, we have obtained two nontrivial *rational* expressions (54) and (69) (yielding algebraic values for h_{real} and λ_{real}).

There is no simple 'down-to-earth' Markov partition, symbolic dynamics, or hyperbolic systems interpretation of these rational results: mapping (1) is actually a *measure-preserving* map. The indeterminacy set of this mapping is very small and does not 'proliferate' under successive iterations as one could expect for generic birational transformations. It would be useful to know whether the large set of the algebraic and rational results presented here are a consequence of the measure-preserving property of the mapping or of the favourable behaviour of the indeterminacy set under successive iterations.

valuable discussions on mappings generated by reflections and, especially, measure-preserving mappings.

Appendix A. Dynamical zeta functions versus homogeneous degree generating functions for nongeneric values

We consider here various nongeneric values of the form $(m-1)/(m+3)$ (with $m \geqslant 7$, m odd).

For $\epsilon = \frac{3}{5}$ (corresponding to $m = 7$) the homogeneous generating function defined in section 3, is

$$G_{3/5}^{\text{Hom}}(t) = 1 + 2t + 4t^2 + 7t^3 + 12t^4$$
$$+ 20t^5 + 33t^6 + 54t^7 + 88t^8 + 142t^9 + 228t^{10} + 366t^{11} + \cdots$$

which is compatible with the expansion of the rational expression

$$G_{3/5}^{\text{Hom}}(t) = \frac{1 - t^{10}}{(1 - t) \cdot (1 - t - t^2 + t^9)}.$$

Recalling a possible rational expression for the corresponding dynamical zeta function [3]

$$\zeta_{3/5}(t) = \frac{1 - t^2}{1 - t - t^2 + t^9} \tag{70}$$

one immediately verifies that the 'universal' relation (20) actually holds.

For $\epsilon = \frac{2}{3}$ (corresponding to $m = 9$) the homogeneous generating function defined in section 3, is

$$G_{2/3}^{\text{Hom}}(t) = 1 + 2t + 4t^2 + 7t^3 + 12t^4 + 20t^5$$
$$+ 33t^6 + 54t^7 + 88t^8 + 143t^9 + 232t^{10} + 375t^{11} + 605t^{12} + \cdots.$$

This could be the expansion, up to order 12, of the simple rational expression

$$G_{2/3}^{\text{Hom}}(t) = \frac{1 - t^{12}}{(1 - t) \cdot (1 - t - t^2 + t^{11})}. \tag{71}$$

These results should be compared with the expansion of the dynamical zeta function. Unfortunately, here, the series for the dynamical zeta function are not sufficiently large to allow any 'safe conjecture'. A possible exact expression does not seem to be equal to $(1 - t^2)/(1 - t - t^2 + t^{11})$, but could be [3]

$$\zeta_{2/3}(t) = \frac{1 - t^2 - t^{11} - t^{12} - t^{13}}{1 - t - t^2 + t^{11}} \qquad \text{or} \qquad \frac{1 - t^2 - t^{11} - t^{12}}{1 - t - t^2 + t^{11}}. \tag{72}$$

The 'universal' relation (20) is verified with (71) together with $(1 - t^2)/(1 - t - t^2 + t^{11})$, *but not with* (71) together with (72). One can, however, imagine that the 'universal' relation (20) could be slightly modified on some of these $(m-1)/(m+3)$ values ($m = 9, 13, \ldots$). For instance, (71) and (72) verify (up to order 12) the simple relation

$$t \cdot \zeta_{2/3}(t) - (1 - t) \cdot G_{2/3}^{\text{Hom}}(t) + 1 - t^{m+2} - t^{m+3} = 0 \qquad \text{where} \quad m = 9. \tag{73}$$

These calculations need to be revisited.

For $\epsilon = \frac{5}{7}$ (corresponding to $m = 11$) the homogeneous generating function defined in section 3 is

$$G_{5/7}^{\text{Hom}}(t) = 1 + 2t + 4t^2 + 7t^3 + 12t^4 + 20t^5$$
$$+ 33t^6 + 54t^7 + 88t^8 + 143t^9 + 232t^{10} + 376t^{11} + 609t^{12} \cdots.$$

This could be the expansion of

$$G_{5/7}^{\text{Hom}}(t) = \frac{1 - t^{14}}{(1 - t) \cdot (1 - t - t^2 + t^{13})}.$$

For $\epsilon = \frac{3}{4}$ (corresponding to $m = 13$) the homogeneous generating function is

$$G_{3/4}^{\text{Hom}}(t) = 1 + 2t + 4t^2 + 7t^3 + 12t^4 + 20t^5$$
$$+ 33t^6 + 54t^7 + 88t^8 + 143t^9 + 232t^{10} + 376t^{11} + 609t^{12} + \cdots.$$

This series is not large enough. It could be the expansion of the simple expression

$$G_{3/4}^{\text{Hom}}(t) = \frac{1 - t^{16}}{(1 - t) \cdot (1 - t - t^2 + t^{15})}.$$

Appendix B. Number of real fixed points of the P-type, Q-type and R-type

Let us just give the number of real nth cycles of the P-type, Q-type and R-type for miscellaneous values of ϵ in increasing order.

For $\epsilon < 0$ (and $\epsilon \neq -1$) one gets the results presented in table B.1.

For $\epsilon = \frac{11}{100}, \frac{1}{4}, \frac{52}{100}, \frac{9}{10}, \frac{11}{10}, 5, 10, 50, 100$, and $20\,000$, one gets the results presented in tables B.2–11.

Table B.1. Number of *real* nth cycles of the P-type, Q-type and R-type for $\epsilon < 0$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
P_n	1	0	1	1	2	2	4	4	6	8	12	12	20	24	30	38	54	65
Q_n	0	0	0	0	0	0	0	1	0	1	0	3	0	4	0	9	0	13
R_n	0	0	0	0	0	0	0	0	2	2	6	10	20	30	60	88	156	238
T_n	1	0	1	1	2	2	4	5	8	11	18	25	40	58	90	135	210	316

Table B.2. Number of *real* nth cycles of the P-type, Q-type and R-type for $\epsilon = \frac{11}{100}$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
P_n	1	0	1	1	2	1	2	1	4	5	8	5	10	11	14	14	20	21
Q_n	0	0	0	0	0	0	0	0	0	1	0	3	0	3	0	5	0	5
R_n	0	0	0	0	0	0	0	0	0	0	*	*	*	*	*	*	*	*
T_n	1	0	1	1	2	1	2	1	4	6	*	*	*	*	*	*	*	*

Table B.3. Number of *real* nth cycles of the P-type, Q-type and R-type for $\epsilon = \frac{1}{4}$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13
P_n	1	0	1	1	2	0	1	0	3	1	4	2	8
Q_n	0	0	0	0	0	0	0	1	0	1	0	2	0
R_n	0	0	0	0	0	0	0	0	0	0	0	0	0
T_n	1	0	1	1	2	0	1	1	3	2	4	4	8

Table B.4. Number of *real n*th cycles of the P-type, Q-type and R-type for $\epsilon = \frac{52}{100}$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
P_n	1	0	1	1	2	0	2	0	4	1	4	2	6	2	8
Q_n	0	0	0	0	0	0	0	1	0	1	0	2	0	4	0
R_n	0	0	0	0	0	0	0	0	0	0	2	8	10	*	*
T_n	1	0	1	1	2	0	2	1	4	2	6	12	16	*	*

Table B.5. Number of *real n*th cycles of the P-type, Q-type and R-type for $\epsilon = \frac{9}{10}$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
P_n	1	0	1	1	0	0	2	0	0	1	4	0	2	1	6	1	6	3
Q_n	0	0	0	0	0	0	0	1	0	1	0	2	0	2	0	4	0	5
R_n	0	0	0	0	0	0	0	0	0	0	*	*	*	*	*	*	*	*
T_n	1	0	1	1	0	0	2	1	0	2	*	*	*	*	*	*	*	*

Table B.6. Number of *real n*th cycles of the P-type, Q-type and R-type for $\epsilon = \frac{11}{10}$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
P_n	1	0	1	0	0	0	2	0	0	1	2	0	2	1	2	1	4
Q_n	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0
R_n	0	0	0	0	0	0	0	0	0	0	0	*	*	*	*	*	*
T_n	1	0	1	0	0	0	2	0	0	2	2	*	*	*	*	*	*

Table B.7. Number of *real n*th cycles of the P-type, Q-type and R-type for $\epsilon = 5$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
P_n	1	0	1	0	0	0	0	0	0	0	2	0	0	1	0	0	2	0
Q_n	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
R_n	0	0	0	0	0	0	0	0	0	0	0	*	*	*	*	*	*	*
T_n	1	0	1	0	0	0	0	0	0	0	2	*	*	*	*	*	*	*

Table B.8. Number of *real n*th cycles of the P-type, Q-type and R-type for $\epsilon = 10$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
P_n	1	0	1	0	2	0	0	1	0	0	2	0	2	1	0	0	2	1
Q_n	0	0	0	0	0	0	0	1	0	0	0	0	0	1	0	0	0	1
R_n	0	0	0	0	0	0	0	0	0	0	0	*	*	*	*	*	*	*
T_n	1	0	1	0	2	0	0	2	0	0	2	*	*	*	*	*	*	*

Table B.9. Number of *real n*th cycles of the P-type, Q-type and R-type for $\epsilon = 50$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
P_n	1	0	1	0	2	0	2	1	2	0	4	1	4	1	2	2	6	3
Q_n	0	0	0	0	0	0	0	1	0	1	0	1	0	2	0	3	0	3
R_n	0	0	0	0	0	0	0	0	0	2	0	*	*	*	*	*	*	*
T_n	1	0	1	0	2	0	2	2	2	3	4	*	*	*	*	*	*	*

Table B.10. Number of *real* nth cycles of the P-type, Q-type and R-type for $\epsilon = 100$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
P_n	1	0	1	0	2	0	2	1	2	0	4	1	6	1	6	2	8	3
Q_n	0	0	0	0	0	0	0	1	0	1	0	3	0	2	0	3	0	6
R_n	0	0	0	0	0	0	0	0	0	2	0	*	*	*	*	*	*	*
T_n	1	0	1	0	2	0	2	2	2	3	4	*	*	*	*	*	*	*

Table B.11. Number of *real* nth cycles of the P-type, Q-type and R-type for $\epsilon = 20\,000$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
P_n	1	0	1	0	2	0	2	1	2	0	4	1	6	1	6	2
Q_n	0	0	0	0	0	0	0	1	0	1	0	3	0	4	0	7
R_n	0	0	0	0	0	0	0	0	0	2	0	2	*	*	*	*
T_n	1	0	1	0	2	0	2	2	2	3	4	6	$\geqslant 8$	$\geqslant 9$	$\geqslant 14$	$\geqslant 17$

Appendix C. Expansions of some real dynamical zeta functions

Let us just give some additional expansions for $\zeta_\epsilon^{\text{real}}(t)$ for increasing values of ϵ.

For $\epsilon = \frac{9}{50}$, one obtains the following expansions for $\zeta_\epsilon^{\text{real}}(t)$:

$$\zeta_{9/50}^{\text{real}}(t) = (1/((1-t)(1-t^3)(1-t^4)(1-t^5)^2(1-t^6)$$
$$\times (1-t^7)^2(1-t^8)^3(1-t^9)^4(1-t^{10})^3(1-t^{11})^8)) \cdots$$
$$= 1 + t + t^2 + 2t^3 + 3t^4 + 5t^5 + 7t^6 + 10t^7 + 16t^8 + 24t^9$$
$$+ 34t^{10} + 52t^{11} + \cdots$$

yielding the following 'rough' approximation for h_{real}: $h_{\text{real}} \simeq (52)^{1/11} \simeq 1.432$.

For the 'nongeneric' value $\epsilon = \frac{1}{5}$, $\zeta_{1/5}^{\text{real}}(t)$ reads

$$\zeta_{1/5}^{\text{real}}(t) = (1/((1-t)(1-t^3)(1-t^4)(1-t^5)^2(1-t^6)$$
$$\times (1-t^7)(1-t^9)^4(1-t^{10})^2(1-t^{11})^5(1-t^{12})^4)) \cdots$$
$$= 1 + t + t^2 + 2t^3 + 3t^4 + 5t^5 + 7t^6 + 9t^7 + 12t^8 + 20t^9$$
$$+ 28t^{10} + 39t^{11} + 55t^{12} + \cdots$$

yielding the following 'rough' approximation for h_{real}: $h_{\text{real}} \simeq (55)^{1/12} \simeq 1.3964$. For $\epsilon = \frac{1}{5}$ the previous Q_n and R_n are equal to zero up to order 12. The exponents in (74) are thus the P_n.

For $\epsilon = \frac{31}{125}, \frac{12}{25}, \frac{66}{125}, \frac{2}{3}, \frac{17}{25}, \frac{3}{4}, \frac{3}{2}$, one obtains, respectively, the following expansions for $\zeta_\epsilon^{\text{real}}(t)$:

$$\zeta_{31/125}^{\text{real}}(t) = (1/((1-t)(1-t^3)(1-t^4)(1-t^5)^2(1-t^6)$$
$$\times (1-t^7)^2(1-t^8)(1-t^9)^4(1-t^{10})^5(1-t^{11})^{12})) \cdots$$
$$= 1 + t + t^2 + 2t^3 + 3t^4 + 5t^5 + 7t^6 + 10t^7 + 14t^8 + 22t^9$$
$$+ 34t^{10} + 54t^{11} + \cdots$$

yielding the approximation for h_{real}: $h_{\text{real}} \simeq (54)^{1/11} \simeq 1.437$;

$$\zeta_{12/25}^{\text{real}}(t) = (1/((1-t)(1-t^3)(1-t^4)(1-t^5)^2(1-t^7)^2$$
$$\times (1-t^8)(1-t^9)^6(1-t^{10})^5(1-t^{11})^{10})) \cdots$$
$$= 1 + t + t^2 + 2t^3 + 3t^4 + 5t^5 + 6t^6 + 9t^7 + 13t^8 + 22t^9$$
$$+ 33t^{10} + 49t^{11} + \cdots$$

yielding the following rough approximation for h_{real} for $\epsilon = \frac{12}{25}$: $h_{\text{real}} \simeq (49)^{1/11} \simeq 1.424$ and

$$
\begin{aligned}
\zeta^{\text{real}}_{66/125}(t) &= (1/((1-t)(1-t^3)(1-t^4)(1-t^7)^2(1-t^8) \\
&\quad \times (1-t^9)^4(1-t^{10})^2(1-t^{11})^4)) \cdots \\
&= 1 + t + t^2 + 2t^3 + 3t^4 + 3t^5 + 4t^6 + 7t^7 + 9t^8 + 14t^9 \\
&\quad + 19t^{10} + 27t^{11} + \cdots
\end{aligned}
$$

yielding $h_{\text{real}} \simeq (27)^{1/11} \simeq 1.349$ for $\epsilon = \frac{66}{125}$.

For $\epsilon = \frac{2}{3}$ (that is $(m-1)/(m+3)$ for $m = 9$) the real dynamical zeta function reads

$$
\begin{aligned}
\zeta^{\text{real}}_{2/3}(t) &= (1/((1-t)(1-t^3)(1-t^4)(1-t^7)^2(1-t^8) \\
&\quad \times (1-t^9)^2(1-t^{10})^2(1-t^{11})^4(1-t^{12})^2)) \cdots \\
&= 1 + t + t^2 + 2t^3 + 3t^4 + 3t^5 + 4t^6 + 7t^7 + 9t^8 + 12t^9 + 17t^{10} + 25t^{11} \\
&\quad + 32t^{12} + \cdots
\end{aligned}
\tag{74}
$$

yielding $h_{\text{real}} \simeq (32)^{1/12} \simeq 1.3348$. Let us note that one must be careful converting systematically a series to a rational function (Padé approximation). Up to order 12, expansion (74) is in agreement with the expansion of the following simple rational expression:

$$
\frac{1 + t + t^3 - t^6}{1 - t^2 - 2t^4 + t^5 - t^6} = \frac{(1 + t + t^3 - t^6) \cdot (1 - t)}{1 - t - t^2 + t^3 \cdot (1 - t + t^2)^2}
\tag{75}
$$

which is reminiscent of the exact expression (54). However, one easily finds that the coefficient of t^{54} in (75) becomes negative (the coefficients grow like $\simeq (-1.5252)^N$). Expression (75) *cannot* be the exact expression of a (real) dynamical zeta function.

For $\epsilon = \frac{17}{25}$, the real dynamical zeta function reads

$$
\begin{aligned}
\zeta^{\text{real}}_{17/25}(t) &= \frac{1}{(1-t)(1-t^3)(1-t^4)(1-t^7)^2(1-t^8)(1-t^9)^2(1-t^{10})^2(1-t^{11})^4} \cdots \\
&= 1 + t + t^2 + 2t^3 + 3t^4 + 3t^5 + 4t^6 + 7t^7 + 9t^8 + 12t^9 \\
&\quad + 17t^{10} + 25t^{11} + \cdots
\end{aligned}
$$

yielding $h_{\text{real}} \simeq (25)^{1/11} \simeq 1.3399$.

For the nongeneric value $\epsilon = \frac{3}{4}$ (that is $(m-1)/(m+3)$ for $m = 13$) the real dynamical zeta function reads

$$
\begin{aligned}
\zeta^{\text{real}}_{3/4}(t) &= (1/((1-t)(1-t^3)(1-t^4)(1-t^7)^2(1-t^8) \\
&\quad \times (1-t^9)^3(1-t^{10})(1-t^{11})^2(1-t^{12}))) \cdots \\
&= 1 + t + t^2 + 2t^3 + 3t^4 + 3t^5 + 4t^6 + 7t^7 + 9t^8 + 13t^9 \\
&\quad + 17t^{10} + 23t^{11} + 30t^{12} + \cdots
\end{aligned}
$$

yielding $h_{\text{real}} \simeq (30)^{1/12} \simeq 1.3277$.

Finally, for $\epsilon = \frac{3}{2}$, the real dynamical zeta function reads

$$
\begin{aligned}
\zeta^{\text{real}}_{3/2}(t) &= \frac{1}{(1-t)(1-t^3)(1-t^7)^2(1-t^{10})^2} \cdots \\
&= 1 + t + t^2 + 2t^3 + 2t^4 + 2t^5 + 3t^6 + 5t^7 + 5t^8 + 6t^9 + 10t^{10} + 10t^{11} + \cdots
\end{aligned}
$$

yielding $h_{\text{real}} \simeq (10)^{1/11} \simeq 1.233$.

Appendix D. Expansions of the 'real Arnold complexity' generating functions

We give here a few expansions for the 'real Arnold complexity' generating functions $\mathcal{A}_\epsilon(t)$. Let us first give the expansion of $\mathcal{A}_\epsilon(t)$ corresponding to $\epsilon = \frac{2}{3}$ in order to compare it with (74) and (75):

$$\mathcal{A}_{2/3}(t) = t + t^2 + 2t^3 + 3t^4 + 3t^5 + 6t^6 + 7t^7$$
$$+11t^8 + 14t^9 + 21t^{10} + 29t^{11} + 37t^{12} + 51t^{13} + \cdots$$

yielding the following estimation for $\lambda_{\text{real}} \simeq (51)^{1/13} \simeq 1.3531$ to be compared with $h_{\text{real}} \simeq (32)^{1/12} \simeq 1.3348$ from (74). The coefficients of the expansions of $\zeta_{2/3}(t)$ and $\mathcal{A}_{2/3}(t)$ are very close. Up to order ten, the ratio $\zeta_{2/3}(t)/\mathcal{A}_{2/3}(t)$ coincides with the expansion of

$$\frac{\zeta_{2/3}(t)}{\mathcal{A}_{2/3}(t)} \simeq \frac{1 - t^2}{t} \cdot \frac{1 - t^5}{1 - 2t^4 + t^5}. \tag{76}$$

The expansion of $\mathcal{A}_\epsilon(t)$ corresponding to the nongeneric value $\epsilon = \frac{3}{4}$ (that is $(m - 1)/(m + 3)$ for $m = 13$) reads

$$\mathcal{A}_{3/4}(t) = 1 + t + t^2 + 2t^3 + 3t^4 + 3t^5 + 6t^6 + 7t^7$$
$$+11t^8 + 12t^9 + 21t^{10} + 27 \cdot t^{11} + 36t^{12} + 47t^{13} + \cdots. \tag{77}$$

The expansions of $\zeta_{3/4}(t)$ and $\mathcal{A}_{3/4}(t)$ are again very close. Expansion (77) yields the estimation for $\lambda_{\text{real}} \simeq (47)^{1/13} \simeq 1.3446$ to be compared with $h_{\text{real}} \simeq (30)^{1/12} \simeq 1.3276$ from (76).

Let us now give the expansion of $\mathcal{A}_\epsilon(t)$ corresponding to values very close to the nongeneric value $\frac{1}{4}$, for instance $\epsilon = \frac{99}{400}$ and $\epsilon = \frac{101}{400}$:

$$\mathcal{A}_{99/400}(t) = t + t^2 + 2t^3 + 3t^4 + 5t^5 + 6t^6$$
$$+9t^7 + 13t^8 + 22t^9 + 33t^{10} + 47t^{11} + 70t^{12} + 101t^{13} + \cdots$$

and

$$\mathcal{A}_{101/400}(t) = t + t^2 + 2t^3 + 3t^4 + 5t^5 + 6t^6$$
$$+9t^7 + 13t^8 + 22t^9 + 33t^{10} + 47t^{11} + 70t^{12} + 109t^{13} + \cdots.$$

Near the nongeneric value $\epsilon = \frac{1}{5}$, for instance for $\epsilon = \frac{99}{500}$ and $\epsilon = \frac{101}{500}$, one gets

$$\mathcal{A}_{99/500}(t) = t + t^2 + 2t^3 + 3t^4 + 5t^5 + 6t^6$$
$$+9t^7 + 13t^8 + 18t^9 + 27t^{10} + 37t^{11} + 62t^{12} + 89t^{13} + \cdots$$

and

$$\mathcal{A}_{101/500}(t) = t + t^2 + 2t^3 + 3t^4 + 5t^5 + 6t^6$$
$$+9t^7 + 13t^8 + 18t^9 + 27t^{10} + 41t^{11} + 66t^{12} + 85t^{13} + \cdots.$$

Near the nongeneric value $\epsilon = \frac{3}{5}$, for instance for $\epsilon = \frac{299}{500}$ and $\epsilon = \frac{301}{500}$, one gets the same expansions:

$$\mathcal{A}_{299/500}(t) = \mathcal{A}_{301/500}(t) = t + t^2 + 2t^3 + 3t^4 + 3t^5 + 6t^6$$
$$+11t^7 + 11t^8 + 16t^9 + 21t^{10} + 29t^{11} + 42t^{12} + 57t^{13} + \cdots.$$

For the nongeneric value $\epsilon = \frac{1}{10}$ one obtains the following result:

$$\mathcal{A}_{1/10}(t) = t + t^2 + 2t^3 + t^4 + 3t^5 + 8t^6$$
$$+9t^7 + 11t^8 + 16t^9 + 21t^{10} + 31t^{11} + 48t^{12} + 58t^{13} + \cdots.$$

We finally give some results for $\epsilon > 1$ (again obtained with 6000 digits). Let us give the expansion of $\mathcal{A}_\epsilon(t)$ for $\epsilon = \frac{3}{2}$, for ϵ near $\epsilon = 2$, for instance for $\epsilon = \frac{2001}{1000}$ or $\epsilon = \frac{1999}{1000}$ and for $\epsilon = 4, 5, 6, 10, 20, 30, 40, 50, 100, 500$ and 1000 respectively:

$$\mathcal{A}_{3/2}(t) = t + t^2 + 2t^3 + t^4 + 3t^5 + 2t^6 + 3t^7 + 3t^8 + 2t^9 + 3t^{10}$$
$$+ 3t^{11} + 4t^{12} + 3t^{13} + \cdots$$

$$\mathcal{A}_{\epsilon \simeq 2}(t) = t + t^2 + 2t^3 + t^4 + t^5 + 2t^6 + t^7 + 3t^8 + 2t^9 + t^{10}$$
$$+ 3t^{11} + 2t^{12} + 3t^{13} + \cdots$$

$$\mathcal{A}_4(t) = t + t^2 + 2t^3 + t^4 + t^5 + 2t^6 + t^7 + t^8 + 2t^9 + 3t^{10}$$
$$+ t^{11} + 2t^{12} + 3t^{13} + t^{14} + \cdots$$

$$\mathcal{A}_5(t) = t + t^2 + 2t^3 + t^4 + t^5 + 2t^6 + 3t^7 + t^8 + 2t^9 + 3t^{10}$$
$$+ t^{11} + 2t^{12} + 3t^{13} + 3t^{14} + \cdots$$

$$\mathcal{A}_6(t) = t + t^2 + 2t^3 + 3t^4 + t^5 + 2t^6 + 3t^7 + 3t^8 + 2t^9 + 3t^{10}$$
$$+ 3t^{11} + 4t^{12} + 3t^{13} + \cdots$$

$$\mathcal{A}_{10}(t) = t + t^2 + 2t^3 + 3t^4 + 3t^5 + 2t^6 + 3t^7 + 3t^8 + 4t^9 + 5t^{10}$$
$$+ 5t^{11} + 4t^{12} + 5t^{13} + \cdots$$

$$\mathcal{A}_{20}(t) = t + t^2 + 2t^3 + 3t^4 + 5t^5 + 4t^6 + 5t^7 + 3t^8 + 8t^9 + 11t^{10}$$
$$+ 7t^{11} + 10t^{12} + 21t^{13} + \cdots$$

$$\mathcal{A}_{30}(t) = t + t^2 + 2t^3 + 3t^4 + 5t^5 + 4t^6 + 5t^7 + 9t^8 + 8t^9 + 11t^{10}$$
$$+ 11t^{11} + 14t^{12} + 25t^{13} + \cdots$$

$$\mathcal{A}_{40}(t) = t + t^2 + 2t^3 + 3t^4 + 5t^5 + 4t^6 + 5t^7 + 9t^8 + 10t^9 + 15t^{10}$$
$$+ 11t^{11} + 14t^{12} + 29t^{13} + \cdots$$

$$\mathcal{A}_{50}(t) = t + t^2 + 2t^3 + 3t^4 + 5t^5 + 4t^6 + 7t^7 + 9t^8 + 10t^9 + 15t^{10}$$
$$+ 17t^{11} + 22t^{12} + 37t^{13} + \cdots$$

$$\mathcal{A}_{100}(t) = t + t^2 + 2t^3 + 3t^4 + 5t^5 + 8t^6 + 7t^7 + 9t^8 + 16t^9 + 19t^{10}$$
$$+ 29t^{11} + 36t^{12} + 51t^{13} + \cdots$$

$$\mathcal{A}_{500}(t) = t + t^2 + 2t^3 + 3t^4 + 5t^5 + 8t^6 + 11t^7 + 17t^8 + 24t^9 + 35t^{10}$$
$$+ 47t^{11} + 64t^{12} + 93t^{13} + \cdots$$

$$\mathcal{A}_{1000}(t) = t + t^2 + 2t^3 + 3t^4 + 5t^5 + 8t^6 + 11t^7 + 17t^8 + 24t^9 + 35t^{10}$$
$$+ 51t^{11} + 72t^{12} + 101t^{13} + \cdots.$$

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